INTERMEDIATE ALGEBRA

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INTERMEDIATE ALGEBRA

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THE MACMILLAN COMPANY • NEW YORK • 1947

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This text is designed for students who, for one reason or another, such as inadequate previous training or merely the immaturity of youth, must learn algebra at a somewhat slower pace than that set by the typical college freshman who plans to enter a scientific, engineering, or allied field. In other words, it is intended to be a true *intermediate* algebra, suitable for college classes catering to those students who in high school had less than the normal amount of mathematics. As such it starts at "bedrock" in algebra, moves on with a leisurely but accelerating pace, and leaves the student at the ledge in the cliff of learning marked roughly by progressions and logarithms. There is ample material for a course of three or six semester hours, according to the needs and proficiency of the class.

Specific features of the text include the following.

1. The illustrative examples are numerous, and they are carefully selected to cover typical cases.

2. The fact that in algebra the rules of arithmetic are applied to letters instead of numbers is emphasized by frequent use of two illustrative examples, one of which is purely arithmetic. Thus the student is encouraged to check methods, rules, and results by use of simple numbers.

3. Common errors are pointed out and discussed before they are hidden as "traps" in problems. In fact, in certain problems the sole question to be considered is whether a given operation is or is not correct.

4. In a text of this level the primary aim is clarity, to-

gether with as much mathematical rigor as can be preserved in the light of the first requirement. A major goal, therefore, has been simplicity and directness of style. To this end exceptions and qualifying statements have been for the most part relegated to footnotes.

5. More than 2600 problems provide ample drill.

6. Experience has indicated that the common practice of printing only half of the answers in a text in effect cuts down the available problem list by almost one half, since most teachers assign for home work only those problems for which the answers are to be found in the text. On the other hand, some problems without answers are needed for correspondence courses, and students acquire self-reliance by learning to check results for themselves. These two conflicting considerations make it seem advisable to include most, but not all, of the answers; and so four fifths of them are included in this text. We think it inadvisable to have the remaining answers printed even in pamphlet form, since copies would eventually reach unintended hands and destroy the effectiveness of the book as a text for correspondence courses.

7. Our guiding principle has been to maintain, so far as the subject matter allows, a consistent, upward-sloping level of difficulty. Such a policy dictates, for example, an early treatment of linear equations, and the deferment of the chapter on exponents and radicals as long as seems expedient.

8. The text has been used in mimeographed form in the appropriate classes at Texas Agricultural and Mechanical College, and the present form embodies changes made in the light of student reactions. It has since been read critically by several referees, who have contributed many valuable suggestions.

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INTERMEDIATE ALGEBRA

Chapter One

ALGEBRA AS A LANGUAGE AND A TOOL

1.1. *The relation between algebra and English*

As applied, algebra is a method of solving problems by the use of numbers, letters, and symbols.

As written and as read, algebra is a brief and useful international language. One must learn to read it, at least in part, before the calculations can be started. Fortunately for the beginning student, many of its symbols are the familiar ones of arithmetic. The others may be learned as they occur in the text. For reference they are grouped together in Table 1.

This language of algebra is usually much briefer than English. For example, the English phrase, "the number which is 3 less than 4 times the number x," becomes, in algebra, simply "$4x - 3$." Again, the statement that "twice a number x, plus 5, equals one-third of the sum of x and 4" is written in algebra as the *equation*

$$2x + 5 = \frac{x + 4}{3}. \tag{1}$$

When certain numbers are always related in a given manner, this fact is expressed conveniently in an algebraic *formula*. For instance, the distance, d, traveled by a body moving at uniform speed equals the product of its rate, r, and the time, t. Or the area, A, of a rectangle is equal to the product of its length, L, and its width, W. These facts lead to the following formulas

$$d = rt; \tag{2}$$

$$A = LW. \tag{3}$$

The problems in Exercise 1 test the student's ability to translate English phrases or sentences into algebraic language.

EXERCISE 1

If x represents any number which we wish to think about, express in terms of x the numbers described in problems 1–6.

1. The double of x.

2. The number which is 4 less than x.

3. Two more than one-half of x.

4. One less than 3 times x.

5. One-half of the number which is 2 less than x.

6. Three times the product of x by its double.

7. A man walks toward a point 30 miles away at 3 m.p.h. How far has he traveled, and how far has he to go, after x hours?

8. If apples and oranges sell for 3¢ and 4¢ each, respectively, what is the selling price of x apples and y oranges?

9. A and B, 20 miles apart, walk toward each other. After A has walked x miles and B has walked y miles, how far apart are they?

10. Work problem 9 if A and B go in the same direction, with A behind B.

Write as formulas the facts stated in problems 11–14.

11. The area, P, of a parallelogram equals the product of its base, B, by its altitude, H.

12. The area, T, of a triangle equals one-half the product of its base, B, by its altitude, H.

13. The circumference, C, of a circle equals twice the product of its radius, R, by the constant π (pronounced "pie"), which is nearly, though not exactly, equal to 3.1416.

14. Simple interest, I, is the product of the principal, P, by the rate, r, by the time, t.

Change the statements in problems 15–18 *to equations in algebra.*

15. Twice the number x, plus 5, equals one-third of the sum of x and 4.

16. Three more than twice x is 2 less than 3 times x.

17. Half the sum of x and twice y is twice the sum of y and thrice x.

18. Two less than twice the sum of x and 1 is 3 more than half the number which exceeds x by 4.

Represent in terms of x *the pairs of numbers described in problems* 19–23.

19. Their sum is 10. (*Answer:* x and $10 - x$.)

20. Their difference is 5.

21. One is one-third of the other.

22. One is 3 more than twice the other.

23. One is 2 less than two-thirds of the other.

24. A boy with x dimes has 3 times as many nickels as dimes. Find the value in cents of what he has.

25. If a man is x years old now, how old was he 5 years ago? How old will he be in 10 years?

1.2. *The relation between algebra and arithmetic*

We have seen that algebra is a method of solving problems by the use of letters which stand for numbers. Arithmetic, on the other hand, is the science of computing with particular numbers. The result of an algebraic problem often can be applied to many cases simply by assigning different values to the letters, whereas an arithmetic result applies only to the one problem concerned.

1.3. *Numbers*

The concept of what is meant by a *number* develops and changes as one studies mathematics. We shall here give illus-

trations of certain types of numbers rather than a definition of numbers in general.

Most easily understood are the *whole numbers*, or *integers*, such as 1, 2, 3, etc. Then there are *fractions*, such as $\frac{1}{2}$, $\frac{3}{5}$, $\frac{9}{4}$, etc. Both integers and fractions may be *positive*, like those above, or *negative*, as -3, -11, $-\frac{2}{3}$, etc. Negative numbers are useful in various situations, as when we say that the temperature is $-20°$ (twenty degrees below zero) or that a river level is -2 feet, meaning that its surface is 2 feet below normal.

Numbers are either *real*, like those mentioned above, or *imaginary*. The second group will be discussed later.

All real numbers may be represented conveniently as points on a line, as shown (in part) in Fig. 1.

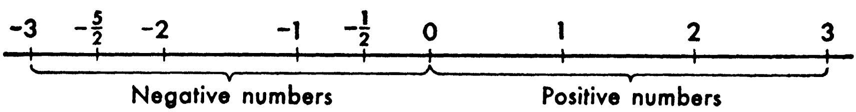

Fig. 1

This figure also calls attention to the very special number *zero*, written 0, which is between the positive and negative numbers and may be classed with either group. That is, $0 = +0 = -0$. This fact makes necessary special rules for computation with zero. (Art. 1.6.)

1.4. *Absolute and numerical values*

The symbol $|a|$, read "the absolute value of a," or $|-a|$, read "the absolute value of $-a$," means the positive number equal to $+a$ if a is positive, or to $-a$, if a is negative. Thus, $|3| = |-3| = 3$.

Consider two numbers: a and b. If a has a larger absolute value than b, it is said to be larger than b *numerically*. If a comes to the right of b in Fig. 1, it is larger than b *algebraically*. Or, to state it another way, a exceeds b algebraically whenever $a - b$ is positive. Thus, 10 is larger than -3 both

numerically and algebraically; while 2 exceeds -3 algebraically but not numerically. The symbols $>$ and $<$ mean respectively "greater than" and "less than" in the algebraic sense. Thus, $0 > -2$ (zero *is greater* than minus two) and $-5 < 2$ (minus five is *less than* two).

1.5. *Operations with positive and negative numbers*

When negative numbers are taken into account the ordinary operations of arithmetic must be explained anew. The rules for addition and subtraction are as follows:

(1) **To add two numbers with like signs (both positive or both negative) add their absolute values and prefix their common sign.**

(2) **To add two numbers with unlike signs, subtract the smaller absolute value from the larger one and prefix the sign of the numerically larger number. If the numbers are numerically equal, their sum is zero.**

(3) **To subtract one number from another, change the sign of the number subtracted and then add.**

These operations are illustrated in column form in the examples below.

Addition

(a) $\begin{array}{r} 2 \\ 5 \\ \hline 7 \end{array}$ (b) $\begin{array}{r} 2 \\ -5 \\ \hline -3 \end{array}$ (c) $\begin{array}{r} -2 \\ 5 \\ \hline 3 \end{array}$ (d) $\begin{array}{r} -2 \\ -5 \\ \hline -7 \end{array}$

Subtraction

(e) $\begin{array}{r} 2 \\ 5 \\ \hline -3 \end{array}$ (f) $\begin{array}{r} 2 \\ -5 \\ \hline 7 \end{array}$ (g) $\begin{array}{r} -2 \\ 5 \\ \hline -7 \end{array}$ (h) $\begin{array}{r} -2 \\ -5 \\ \hline 3 \end{array}$

The operations above may be written on single lines by observing the rule that when parentheses are removed the signs enclosed are changed if and only if the sign before the parentheses is minus. Thus $-(+6) = -6$; $-(-6) = +6$ (or

just 6); $+(+6) = 6$; and $+(-6) = -6$. Accordingly, (**d**) and (**h**), for example, become

(**d**) $$(-2) + (-5) = -2 - 5 = -7;$$
(**h**) $$(-2) - (-5) = -2 + 5 = 3.$$

For multiplication and division of positive and negative numbers a single rule suffices:

(4) **The product, and also the quotient, of two numbers with like signs is positive; of two numbers with unlike signs it is negative.**

Thus, $(+4)(+5) = 4 \cdot 5 = 20$; $(-4)(-5) = 20$; $(+4)(-5) = -20$; and $(-4)(+5) = -20$. Also, $10 \div 2 = 5$; $(-10) \div (-2) = 5$; $10 \div (-2) = -5$; and $(-10) \div 2 = -5$.

As shown in the above example, multiplication of positive numbers may be indicated by dots instead of parentheses, provided that the dots are raised to avoid confusing them with decimal points. In the case of letters representing numbers, the omission of a sign between them indicates multiplication, so that, for example, ab means "a times b." This practice, of course, would not do for numbers themselves. Why?

Division is indicated by use of a horizontal bar, or the symbol $\div$. For example, $10 \div 5 = \frac{10}{5} = 2$.

1.6. *Operations with zero*

The rules for operations with zero are illustrated in equations (1) to (5) below.

(1) $0 + 6 = 6 + 0 = 6$; $0 - 7 = -7 + 0 = -7$; $0 + 0 = 0$.
(2) $5 - 0 = -0 + 5 = 5$; $-3 - 0 = -0 - 3 = -3$; $0 - 0 = 0$.

The value of a number is unchanged when zero is added to it or subtracted from it.

(3) $$0 \cdot 4 = 4 \cdot 0 = 0 \cdot (-4) = (-4) \cdot 0 = 0.$$

The product of any number and zero is zero.

$$(4) \qquad \frac{0}{7} = \frac{0}{-7} = 0.$$

The quotient obtained by dividing zero by any number other than zero is zero.

$$(5) \qquad \frac{6}{0} \text{ and } \frac{-6}{0} \textit{ are not solvable.}$$

That is, *it is impossible to divide any number by zero.*

To see that the statements in (5) are reasonable, notice that the value of $\frac{6}{2}$, or 3, is "the number of 2's which will add up to 6." Similarly, $\frac{6}{0}$ asks "how many zeros will add up to 6?" * Evidently no number meets this condition, since the sum of any number of zeros is zero. Similarly, if $\frac{6}{0}$ = (some definite number), then 0 times that number should equal 6; but this is impossible, as seen from equation (3).

EXERCISE 2 (ORAL)

1. Arrange the following numbers so that, as read from left to right, they increase algebraically: -1, -15, 0, 6, $\frac{2}{3}$, -4, -5, $-\frac{1}{2}$, 5, -2.

2. Arrange the numbers of problem 1 so that, as read from left to right, they increase numerically, or in absolute values.

Perform the additions and subtractions as indicated in problems 3–14.

3. $3 + (-7)$.	**4.** $(-6) + 3$.	**5.** $-8 + (-5)$.
6. $-15 + (+5)$.	**7.** $-9 - (-2)$.	**8.** $8 - (-4)$.
9. $6 - 11$.	**10.** $(13) - (8)$.	**11.** $-7 - 0$.
12. $0 + (-17)$.	**13.** $0 - (-32)$.	**14.** $-15 - (-0)$.

* It might be thought that by this test the value of $\frac{0}{0}$ must be one, since "one zero adds up to zero"; but so do two zeros, three zeros, etc. The rule stated in (5) holds in *all* cases.

Perform the additions as indicated in problems 15–26.

15. $\begin{array}{r} 5 \\ \underline{8} \end{array}$ **16.** $\begin{array}{r} -8 \\ \underline{4} \end{array}$ **17.** $\begin{array}{r} -8 \\ \underline{-1} \end{array}$ **18.** $\begin{array}{r} 1 \\ \underline{-8} \end{array}$

19. $\begin{array}{r} -5 \\ \underline{8} \end{array}$ **20.** $\begin{array}{r} -2 \\ \underline{-6} \end{array}$ **21.** $\begin{array}{r} 3 \\ \underline{0} \end{array}$ **22.** $\begin{array}{r} 0 \\ \underline{-5} \end{array}$

23. $\begin{array}{r} 0 \\ \underline{10} \end{array}$ **24.** $\begin{array}{r} -17 \\ \underline{0} \end{array}$ **25.** $\begin{array}{r} 0 \\ \underline{-20} \end{array}$ **26.** $\begin{array}{r} 31 \\ \underline{0} \end{array}$

27–38. *Subtract the lower from the upper numbers in problems* 15–26.

39–50. *Multiply the numbers in problems* 15–26.

Perform the indicated multiplications in problems 51–62.

51. $6(-7)$. **52.** $(-15)(0)$. **53.** $0(100)$.
54. $9(-4)(-5)$. **55.** $(-0)(28)$. **56.** $(-5)(6)$.
57. $+7(-3)$. **58.** $-(-1)(-1)$. **59.** $4(-1)$.
60. $-(2)(-3)$. **61.** $-0(-15)$. **62.** $(+7)(-2)$.

Express as integers the quotients indicated in problems 63–71.

63. $20 \div 4$. **64.** $(-21) \div (7)$. **65.** $14 \div (-2)$. **66.** $(-20) \div 2$.

67. $\frac{12}{-6}$. **68.** $\frac{12}{2}$. **69.** $\frac{-15}{3}$. **70.** $\frac{6}{-3}$. **71.** $\frac{-25}{-5}$.

Which of the quotients indicated in problems 72–87 *are true numbers? Find the values of these numbers.*

72. $\frac{0}{17}$. **73.** $\frac{0}{-25}$. **74.** $\frac{13}{-0}$. **75.** $\frac{-26}{0}$.

76. $\frac{-0}{14}$. **77.** $\frac{-0}{-1}$. **78.** $\frac{0}{1}$. **79.** $\frac{0}{0}$.

80. $\frac{0-4}{4}$. **81.** $\frac{0(-4)}{0-4}$. **82.** $\frac{0-8}{0(-8)}$. **83.** $\frac{4+0}{4-0}$.

84. $\frac{0-4}{0-2}$. **85.** $\frac{4(+0)}{4-0}$. **86.** $\frac{-6+0}{-6-0}$. **87.** $\frac{-6(+0)}{-6+0}$.

1.7. *Algebraic expressions*

A single number or letter, or a group of them, is called an *algebraic expression.*

Examples. $4; x; 3x; 2x - 1; \frac{1}{x}; \frac{2-x}{3}.$

An expression with no plus or minus or equality signs between its parts is called a *term.*

Examples. $7; -2; a; xy; xyz; \frac{2}{x}; -2ab^3.$

The term $-2ab^3$ stands for "-2 times a times b^3," where -2 is the *numerical coefficient,* a and b are *literal* numbers, or letters standing for numbers, and b^3 (read "b cubed" or "the third power of b") means bbb, or *b taken 3 times as a factor.* Here b is the *base* and 3 is the *exponent* of b. Thus, if $a = 3$ and $b = 2$, then $-2ab^3 = -2 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = -48$.

Terms differing only in numerical coefficients, such as $3x^2$ and $-2x^2$, or as^3, $4as^3$, and $-3as^3$, are called *like* terms. An expression of one term is called a *monomial;* of two terms, a *binomial;* of three terms, a *trinomial;* and of two or more terms, a *polynomial.* The terms of a polynomial are connected by plus or minus signs, as in the example:

$$2x^5 - 3x^4 + 5x^2 - 3ax + 2a + b.$$

1.8. *Addition and subtraction of polynomials*

Here it is convenient to state in the form of *laws* certain unproved but reasonable assumptions about numbers represented by letters. Such assumptions are called postulates.

(1) **The commutative law of addition: The sum of two or more algebraic terms is not changed by changing their order.**

Thus, just as $3 + 4 = 4 + 3$, so $x + y = y + x$, no matter what number values we assign to x and y.

(2) **The associative law of addition: The sum of any number of algebraic terms is the same, however we group them.**

That is, just as $4+9+5=(4+9)+5=4+(9+5)=18$, so, in general, $x + y + z = (x + y) + z = x + (y + z)$.

(3) **The distributive law: The sum of the products of one term by each of several other terms is the product of this term by the sum of the others.**

That is, $ab + ac + ad = a(b + c + d)$. The student should verify this equation for various special values of the letters, such as $a = 2$, $b = 3$, etc. Once granted, it enables us to combine the like terms of a polynomial. Thus, $2x^3 + 5x^3 = x^3(2+5) = 7x^3$, and $ax^2+7ax^2-2ax^2=ax^2(1+7-2)=6ax^2$.

We are now in position to add or subtract polynomials by combining like terms. For example,

$$
\begin{aligned}
&(x^3 - 2x^2 + x + 7) + (3x + 4x^3 - 5 + 6x^2)\\
&\quad = (x^3 - 2x^2 + x + 7 + 3x + 4x^3 - 5 + 6x^2) && \text{(by the associative law)}\\
&\quad = (x^3 + 4x^3 - 2x^2 + 6x^2 + x + 3x + 7 - 5) && \text{(by the commutative law)}\\
&\quad = (x^3 + 4x^3) + (-2x^2 + 6x^2) + (x + 3x) + (7 - 5) && \text{(by the associative law)}\\
&\quad = 5x^3 + 4x^2 + 4x + 2. && \text{(by the distributive law)}
\end{aligned}
$$

In subtracting one polynomial from another we note that $-(a+b-c) = (-1)(a+b-c) = (-1)a+(-1)b+(-1)(-c)$ (by the distributive law) $= -a - b + c$ (by the rule of signs in multiplication). This indicates that *when parentheses preceded by a minus sign are removed, the signs of all terms which had been inside are changed*. Thus,

$$
\begin{aligned}
&(x^3 - 2x^2 + x + 7) - (3x + 4x^3 - 5 + 6x^2)\\
&\quad = x^3 - 2x^2 + x + 7 - 3x - 4x^3 + 5 - 6x^2\\
&\quad = -3x^3 - 8x^2 - 2x + 12.
\end{aligned}
$$

The work is shown in column form below. For convenience the terms are arranged in the order of descending powers of x.

Addition	*Subtraction*
$x^3 - 2x^2 + x + 7$	$x^3 - 2x^2 + x + 7$
$4x^3 + 6x^2 + 3x - 5$	$4x^3 + 6x^2 + 3x - 5$
$5x^3 + 4x^2 + 4x + 2$	$-3x^3 - 8x^2 - 2x + 12$

1.9. *Removal and introduction of symbols of aggregation*

We have seen that the signs of terms within parentheses preceded by a minus sign were changed when the parentheses were removed, but were not changed if the preceding sign was plus. Like statements are true of the other symbols of aggregation (Table 1). This enables us to simplify expressions containing such symbols (parentheses, brackets, and braces) by removing them one pair at a time, beginning with the innermost ones. The procedure is then continued on the new expressions formed until all such symbols have been removed and like terms collected.

Example

$$\begin{aligned} &2x + \{3x - [2x - 5(x - 1)]\} \\ &\quad = 2x + \{3x - [2x - 5x + 5]\} \\ &\quad = 2x + \{3x - 2x + 5x - 5\} \\ &\quad = 2x + 3x - 2x + 5x - 5 \\ &\quad = 8x - 5. \end{aligned}$$

When parentheses, or other symbols of aggregation, are introduced into an expression, the signs of the terms within are not changed if the preceding sign is plus, but *are* changed if the preceding sign is minus. This statement may be verified by removing the symbols in accordance with the preceding rules.

Example 1.

$$\begin{aligned} 3x + 2y + z - w &= (3x + 2y) + (z - w) \\ &= (3x + 2y) - (-z + w). \end{aligned}$$

Example 2.

$$3x - 2y - z = 3x + (-2y - z) = 3x - (2y + z).$$

Example 3.

$$2x - y - (a + b) = 2x + [-y - (a + b)]$$
$$= 2x - [y + (a + b)].$$

It is important to notice that the same terms can be enclosed within parentheses preceded by either a plus or a minus sign. The above examples illustrate how the last two terms in each are enclosed with either sign preceding.

EXERCISE 3

1. Write three like terms involving x, y, and z.

2. Write two examples of monomials; of binomials; of trinomials.

Add the polynomials in problems 3–12.

3. $4x - 7; 2x + 3.$
4. $2x + 3; 4 - 7x.$
5. $2ax^2 + 3x - 2; ax^2 - 5x + 3.$
6. $3ax^2 - 2x + 4; 2ax^2 - 3x + 5.$
7. $x^3 - 5x^2 + ax - 2; 3x^3 + 2x^2 - 3ax + 1.$
8. $x^3 - 2x^2 - 3ax + 1; 2x^3 + 2x^2 - 3ax + 1.$
9. $x - 5x^4 + 3x^2 - 2a; 2x^4 - 2x^2 - 1 + 3x.$
10. $4x + 3x^3 - x^4 + 3; 2x + x^4 + 4a - 3x^2.$
11. $2; x^3 - x + 7; 3x^2 + 4 - 2x.$
12. $x - 2; x^4 - 3x + x^2 - 1; 6 + x^2.$

13–16. Verify the additions in problems 3–6 when x and a are replaced by 1 and 0 respectively.

17–20. Verify the additions in problems 3–6 when x and a are replaced by 2 and -3 respectively.

21. If the arithmetic results obtained in problems 13–20 do not expose any errors in the algebraic sums obtained in problems 3–6, does this prove that these sums are correct?

22–29. Subtract the second from the first polynomial in each of problems 3–10.

30–37. Subtract the first from the second polynomial in each of problems 3–10.

In problems 38–47 remove the symbols of aggregation and simplify.

38. $2a - (b + 3a - d)$.
39. $3x - (2y - 4x + 1)$.
40. $2 - 3(x - y) + 2(x + y)$.
41. $4 + 2(a - 2b) - 3(2a - b)$.
42. $2a + [3a - (2b + 1) - c] + 1$.
43. $3c + [2b - (a - c) + 4] - 3$.
44. $2 - \{3x - [x - (2x - 1) + 2] - 2\}$.
45. $3a - \{2b - [3a - 2(2b - 1)] + 3\}$.
46. $-\{2x^2 - [(3ax + 3b)x + x^2]\}$.
47. $-\{3x^2 - [x - (2ax - 1)x]\}$.

Insert parentheses preceded by a minus sign which will include all but the first term in each of problems 48–51.

48. $2x - 3a + b$.
49. $4x - 2a - 3y$.
50. $a + 2b - c + 2$.
51. $3a + 4b - c - 1$.
52. To the sum of $b - 2c + 3a$ and $-6a - 2b + 3c$ add the result of subtracting $b + 4c$ from $3a - b - 2c$.
53. Subtract $3x^2$ from $5x^3 - 2x^2 + 7x - 2$ and add the result to $-3x^3 + 4x^2 - 1$.
54. Take $2a^2 - 4ab$ from $-5a^2 + 6ab + 3$ and add the remainder to $4a^3 + 2ab - 3a^2$.
55. To what expression must $2a^2 - 3ab + 6$ be added to give 0? 1? -3?

1.10. *Multiplication and division of polynomials*

In addition to the distributive law already discussed, two other useful laws about multiplication may be mentioned here.

(1) **The commutative law of multiplication: The product of two or more factors is not changed by changing their order.**

That is, just as $2 \cdot 3 = 3 \cdot 2$, so $ab = ba$ in all cases.

(2) **The associative law of multiplication: The product of three or more factors is not changed by grouping them in different ways.**

Since $(2 \cdot 3)4 = 2(3 \cdot 4)$, for example, so, in general, we assume that $(ab)c = a(bc)$.

Two of the so-called *laws of exponents*, which are discussed at length in Chapter 7, will be needed here.

(3) $$a^m a^n = a^{m+n}.$$

Examples. $2^2 2^3 = 2^5$; $x^4 x^7 = x^{11}$; $aa^2a^7 = a^1a^2a^7 = a^{10}$.

(4) $$\frac{a^m}{a^n} = a^{m-n} \text{ when } m > n.$$

Examples. $\dfrac{3^5}{3^2} = 3^{5-2} = 3^3$; $\dfrac{a^9}{a^2} = a^7$.

Finally, law (5) below, treated more fully in Chapter 3, is used in the division of polynomials.

(5) $$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}.$$

Example. $\left(\dfrac{2}{5}\right)\left(\dfrac{3}{7}\right) = \dfrac{2 \cdot 3}{5 \cdot 7} = \dfrac{6}{35}$.

Algebraic terms may be multiplied or divided by making use of the five laws above.

Example 1.
$$\begin{aligned}(2ax^3)(3a^2x^4) &= 2 \cdot 3aa^2x^3x^4 \text{ [by (1)]}\\ &= (2 \cdot 3)(aa^2)(x^3x^4) \text{ [by (2)]}\\ &= 6a^3x^7 \text{ [by (3)].}\end{aligned}$$

Example 2.
$$\begin{aligned}\frac{10a^8x^4}{2a^3x} &= \left(\frac{10}{2}\right)\left(\frac{a^8}{a^3}\right)\left(\frac{x^4}{x}\right) \text{ [by (5)]}\\ &= 5a^{8-3}x^{4-1} \text{ [by (4)]} = 5a^5x^3.\end{aligned}$$

The multiplication of two polynomials, such as is indicated in the expression $(x^3 - 2x^4 + x + 3x^2 - 1)(x - 3x^3 + 2)$, can be accomplished conveniently by arranging the polynomials in terms of descending powers of x and then employing law (3), as in the following example.

$$\begin{array}{rrrrrrrrl}
-2x^4 & +\ x^3 & +\ 3x^2 & +\ x & -\ 1 & & & & \text{(first factor)}\\
 & & -\ 3x^3 & +\ x & +\ 2 & & & & \text{(second factor)}\\
\hline
6x^7 & -\ 3x^6 & -\ 9x^5 & -\ 3x^4 & +\ 3x^3 & & & & \\
 & & -\ 2x^5 & +\ x^4 & +\ 3x^3 & +\ x^2 & -\ x & & \\
 & & & -\ 4x^4 & +\ 2x^3 & +\ 6x^2 & +\ 2x & -\ 2 & \\
\hline
6x^7 & -\ 3x^6 & -\ 11x^5 & -\ 6x^4 & +\ 8x^3 & +\ 7x^2 & +\ x & -\ 2 & \text{(product).}
\end{array}$$

An indicated division, such as $\dfrac{(5x^3 - 5x^4 + x^5 + x + 5)}{(x^2 - 3x - 2)}$, may be carried through as here shown. The explanation of the steps may be found just below the actual division.

$$
\begin{array}{r l}
 & x^3-2x^2+x-1 \text{ (Quotient*)} \\
\text{(divisor) } x^2-3x-2 & \overline{\big)\, x^5-5x^4+5x^3 \qquad + x+5} \text{ (dividend)} \\
 & \underline{x^5-3x^4-2x^3} \\
 & \quad -2x^4+7x^3 \qquad + x+5 \\
 & \quad \underline{-2x^4+6x^3+4x^2} \\
 & \qquad\qquad x^3-4x^2+x+5 \\
 & \qquad\qquad \underline{x^3-3x^2-2x} \\
 & \qquad\qquad\quad -x^2+3x+5 \\
 & \qquad\qquad\quad \underline{-x^2+3x+2} \\
 & \qquad\qquad\qquad\qquad 3 \text{ (remainder)}
\end{array}
$$

Explanation. The first step is to arrange the dividend and divisor in the order of the descending powers of the same letter. Here the dividend, divisor, and Quotient, so arranged with respect to the powers of x, are respectively $x^5 - 5x^4 + 5x^3 + x + 5$, $x^2 - 3x - 2$, and $x^3 - 2x^2 + x - 1$. The first term (x^3) of the Quotient is obtained by dividing the first term (x^5) of the dividend by the first term (x^2) of the divisor. The result, (x^3), is then multiplied by the divisor, and the product is subtracted from the dividend. The remainder thus obtained forms the new dividend, $-2x^4 + 7x^3 + x + 5$. To obtain the second term, $(-2x^2)$, of the Quotient, the procedure above is repeated, with x^2 being divided into $-2x^4$. Succeeding terms of the Quotient are thus obtained until either a zero remainder is reached or one in which the exponent of the letter upon which the original arrangement is based (in this case, x) is less than the largest exponent it has in the divisor. The final result shows that

$$\frac{x^5 - 5x^4 + 5x^3 + x + 5}{x^2 - 3x - 2} = x^3 - 2x^2 + x - 1 + \frac{3}{x^2 - 3x - 2}.$$

* The word "quotient" is often used with two different meanings. We shall use it with a small "q" to mean the total result of an indicated division. Thus the "quotient" of 27 by 5 is $5\frac{2}{5}$, while the "Quotient" is 5.

In general, $\dfrac{\text{dividend}}{\text{divisor}} = \text{Quotient} + \dfrac{\text{remainder}}{\text{divisor}}$.

The result of a long division should always be stated in the form indicated above.

EXERCISE 4

Perform the multiplications or divisions as indicated in problems 1–7.

1. $(3x^2a)(4a^3x)$. **2.** $5x^3(-3bx)$. **3.** $(-4x)(2ax)(-3a^2x^3)$.

4. $\dfrac{12a^3y^4}{-2a^2y}$. **5.** $\dfrac{6a^7b^6c^3}{-3a^2bc}$. **6.** $\dfrac{-6xy^2z^3}{2y^2z^3}$. **7.** $\dfrac{-8x^4y^7z^2}{-2x^3y^2z^2}$.

8. Verify that $[(x+1)(x-1)](2x+1) = (x+1)[(x-1)(2x+1)]$.

Multiply the pairs of polynomials in problems 9–21.

9. $(6x^7 + 11x^2 - x - 5); (2x)$.
10. $(3x^5 - 11x^4 + 7x^2 - 5x + 2); (3x)$.
11. $(3x^4 - 7x^3 + 2x^2 - 9x + 3); (3x - 1)$.
12. $(2x^4 - 5x^3 + 5x - 2); (x^2 - 3x + 2)$.
13. $(6x^4 + 7x^3 - 10x^2 - 5x + 2); (2x^2 + 3x - 1)$.
14. $(x^5 + x^4 - 5x^3 + 5x^2 + 4x - 2); (x^2 - 3x + 1)$.
15. $(6x^5 - 5x^3 + 4x + 4x^4 + 5x^2 - 2); (-1 + 2x + 2x^2)$.
16. $(4 + 2x - x^2 - 4x^3 + 6x^4); (x + 3x^2 - 1)$.
17. $(3x^3 + 8x^2 - 3 + 2x); (x^2 - 1 + 2x)$.
18. $(8 - 12x + 4x^3); (3x + 2x^2 - 2)$.
19. $(13x - 6x^2 - 11x^3 + 3x^4); (3x - 1 + x^2)$.
20. $(5x^2 + 1 - 5x + 4x^3); (1 + 4x^2 - 3x)$.
21. $(-x + 6x^3 - 3 + 11x^2); (x - 2 + 3x^2)$.

22–34. Divide the first polynomial by the second one in each of problems 9–21.

35. Divide $3x^5 + 2x^2 - 3x^3 + 1$ by $2x^2 - 3x + 1$.
36. Divide $a^3 + b^3$ by $a + b$.
37. Divide $a^3 - b^3$ by $a - b$.
38. Divide $27ax^2 - 25a^3 + 20a^2x - 18x^3$ by $6x - 5a$.

If $A = 3x^2 - 2x + 4$, $B = 3x^2 - x - 1$, and $C = -3x + 1$, *find the values of the quantities in problems* 39–41.

39. $2A - 3B$. **40.** $BC + AB$. **41.** $(A + 2B) \div C$.

REVIEW EXERCISES

If x *represents a number, write the numbers indicated in problems* 1–7.

1. Four times the number.

2. Three-Fourths of it.

3. 20% of it.

4. The difference between the number and 7. (Two answers.)

5. Eight more than the number.

6. Nine less than three times the number.

7. The excess of 5 over 2.5 times the number.

If x *and* y *represent the tens' digit and the units' digit respectively in a number of two digits, write the numbers indicated in problems* 8–11.

8. The number.

9. The number when the digits are reversed.

10. The sum of the digits divided by 3.

11. Twice the number, increased by 36.

If x *represents the width in yards of a rectangular field, express in terms of* x *the length, perimeter, and area of the field under the conditions stated in problems* 12 *and* 13.

12. The length is four times the width.

13. The length is nine yards less than twice the width.

If A *walks* x *feet per second and* B *walks two feet a second faster, write the quantities described in problems* 14–16.

14. The distance A travels in ten minutes.

15. The distance B travels in half an hour.

16. The distance traveled by both in one hour.

Write as formulas the statements in problems 17–26.

17. The volume, V, of a sphere is equal to the product of $\frac{4}{3}$, π, and the cube of the radius, R.

18. The area, A, of a trapezoid is equal to the product of the altitude, h, by one-half the sum of the parallel sides, a and b.

19. The cost, T, in dollars for a telegram of 50 words is computed at the rate of m cents for each of the first ten words and x cents for each of the others.

20. At p dollars each, x books would cost C dollars.

21. D is the difference between twice x and y. (Two formulas.)

22. A man's age is x now and was y ten years ago.

23. On \$5000 at $x\%$ $\left(x\% = \dfrac{x}{100}\right)$ the yearly interest is I dollars.

24. The yearly interest on R dollars at 7% is I dollars.

25. The cost in cents of s pounds of nuts at 50¢ per pound and y pounds of candy at 75¢ per pound is C.

26. The distance in feet traveled in x hours at the rate of five miles per hour is s.

Evaluate the numerical expressions in problems 27–34.

27. $(-4)(-2)(3)$. **28.** $-(-3)^2$. **29.** -3^2.

30. $[(-1) - (-3)][4 - (-2)]$. **31.** $-20 + 0$.

32. $\dfrac{-20+0}{0(-20)}$. **33.** $\dfrac{20-20}{20+20}$. **34.** $\dfrac{20+20}{20-20}$.

Find the missing number in each of problems 35 *to* 40.

35. $7 + (?) = -11$. **36.** $-8 - (?) = 10$.

37. $(-2) \times (?) = 16$. **38.** $-25 + (?) = 5$.

39. $(?) - 8 = -10$. **40.** $3 \times (?) = -15$.

If $\boldsymbol{a} = 3$, $\boldsymbol{b} = -2$, *and* $\boldsymbol{c} = 1$, *find the values of the expressions in problems* 41–46.

41. $-3a^2 + 2b - c$. **42.** $\dfrac{a+b+c}{2bc}$. **43.** $3a^3c - 5$.

44. $(a + b)(b + c)$. **45.** $a + b(b + c)$. **46.** $\dfrac{-a - b(b - c)}{b + c}$.

Add the expressions in each of problems 47–51.

47. $\begin{array}{r} -3b^2 \\ 2b^2 \\ \underline{-7b^2} \end{array}$

48. $\begin{array}{r} 3y^2 \\ -2y^2 \\ \underline{-y^2} \end{array}$

49. $\begin{array}{r} -3(a+b) \\ -2(a+b) \\ \underline{7(a+b)} \end{array}$

50. $\begin{array}{rrr} 4x^2 & -\ y^2 & +\ z \\ & -\ 2y^2 & -\ 3z \\ \underline{3x^2} & & \underline{+\ 2z} \end{array}$

51. $\begin{array}{r} 2x - \ y + 3 \\ -3x + 3y - 1 \\ \underline{x + \ y - 2} \end{array}$

52. Add: $4a^2 - 3ab + 6c$; $4ab - 2a^2 - 3c$; $-2c - 10\,ab + 6a^2$.

53. Add: $7mn - 3n^2 + 6$; $-5n^2 - 7mn - 2$; $6 - mn$.

Subtract the lower from the upper expression.

54. $\begin{array}{r} 4a - 7b - \ c \\ \underline{-2a + 3b - 2c} \end{array}$

55. $\begin{array}{r} 3x^2 + 4xy - \ y^2 \\ \underline{- 5xy + 2y^2} \end{array}$

56. Subtract $4x - y + 2z$ from $3x + y - 4z$.

57. To the sum of $-4x^2 + 3xy - 2$ and $-5xy - 7 + 8x^2$ add the excess of $4xy - 2$ over $2x^2 - 8xy + 6$.

58. How much must be added to the sum of $3ab + 7c^2 - 2d$ and $-6 + 3c^2$ to give $-5ab - c^2$?

Remove the symbols of aggregation and collect like terms.

59. $4 - [2x - 1 - (3x - 1)]$.

60. $-[(2x^2 - 3y^2) + y] - [-x^2 + (2y - x)]$.

61. $-3(a - 2b) + 6\{-4 - [b - 2(4 + b)]\}$.

62. $(3x + y + 6) - [(2x - y - 4) - (3y - 2)]$.

Insert parentheses preceded by a minus sign which will include all but the first two terms.

63. $-3x^2 - y - 2x + z - 3w$.

64. $a^2 - b^2 + 4a^4 - 2a^2b^2$.

65. $x^2 + 2xy + 3y^2 + 4$.

66. $-5 + y - x - 3z^2$.

Carry out the indicated multiplications.

67. $(-3x)(2y)$.

68. $(-3x^2y)(-xy^2)(7x^4y^3)$.

69. $-(-3mn^2)(-5m^2n)$.

70. $3x^2(2xy - y^2)$.

71. $-5mn(2m^2n - 3m + 2n)$.

72. $(3x^2-y+6x)(-x+y-x^2)$.

73. $\begin{array}{r} 4a - 2b \\ \underline{a + 7b} \end{array}$

74. $\begin{array}{l} x^2 + y^2 \\ \underline{x - y} \end{array}$

75. $\begin{array}{r} 2x^2 - 2xy - y^2 \\ \underline{\qquad\qquad 3x - y} \end{array}$

76. $\begin{array}{l} x^3 + x^2y - 2y \\ \underline{3x^2 - 2xy + y} \end{array}$

77. $(x + 4)(x + 3)$.

78. $(x - 5)(x - 2)$.

79. $(x + 7)(x - 3)$.

80. $-(x + 2)(x - 3)$.

81. $-3(a - 1)(a + 1)$.

82. $(2a + 3)(a - 4)$.

83. $(3a - 2)(2a - 5)$.

84. $-2(5a - 1)(3a + 1)$.

85. $-3(2 - x)(3x - 1)$.

86. $(2x + y)(2x - y)$.

87. $(-3x + y)(y + 3x)$.

88. $-(2x - y)(2x + y)$.

89. $-5(3x^2 - y)(3x^2 + y)$.

90. $(2x^2 + 7)(2x^2 + 7)$.

91. $(3 - 2x)(3 - 2x)$.

92. $-2(4 - x)(4 - x)$.

93. $-3(x + 2)^2$.

94. $(10x + 3)^2$.

95. $-(5x - 4y)^2$.

96. $(4x + 5y)^2$.

Note. When the multiplications in problems 77–96 have been carried out by the methods given in this chapter, the student should study Arts. 2.2 and 2.3 and then do the problems orally.

Perform the indicated divisions.

97. $\dfrac{-64xy^2}{8xy}$.

98. $\dfrac{-125m^3}{-5m^2}$.

99. $18a^2b^4 \div 2ab^2$.

100. $(-12x^2y^3z) \div (-4x^2y^2)$.

101. $(6x^3 - 2x + x^2 + 3) \div (2x - 1)$.

102. $(9x^2 - 24xy + 16y^2) \div (3x - 4y)$.

103. $(3x^2 - 5x + 2x^3 - 1) \div (3x + 1)$.

Chapter Two

TYPE PRODUCTS AND FACTORING

2.1. *The factoring rule*

In arithmetic, since $12 = 2 \cdot 2 \cdot 3 = 4 \cdot 3 = 6 \cdot 2 = 1 \cdot 12$, we say that the positive integral *factors* of 12 are 1, 2, 3, 4, 6, and 12. This means that if we divide the integer 12 by any one of these integers, we get another integer. Of these factors only 2 and 3 are *prime*, or without positive integral factors other than themselves and 1.

In algebra it is possible to make use of a similar factoring rule. To explain it, we shall need some definitions.

A polynomial which is the sum of terms such as $2x^2y^3$, in which all exponents are positive integers, is said to be *integral* and *rational*.* In this chapter we shall deal only with integral rational polynomials, including in this general description not only expressions with two or more terms, such as $2x + 3$ and $3x^2 - 5xy + 4$, but even single terms such as $3x$ and 5.

When a polynomial is expressed as the product of two or more integral rational polynomials, it is said to be *factored*, and the parts which are multiplied together are the *factors*. Thus, when we write $x^2 - 1 = (x - 1)(x + 1)$, and $3x + 6 = 3(x + 2)$, we factor the left sides. Each of the four factors $x - 1$, $x + 1$, 3, and $x + 2$ is *prime*, since it has no factors except itself and 1. In this chapter we shall bar any factoring which introduces coefficients that are fractions or types of numbers not yet discussed in our text. Thus, while it is correct

* The reason for the customary use of these adjectives is suggested in the note at the end of Art. 7.5.

to write $x + 2 = 2(\frac{1}{2}x + 1)$, and $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, where $(\sqrt{2})^2 = 2$, the expressions $x + 2$ and $x^2 - 2$ will be considered here as prime, or non-factorable.

When we *completely factor* a polynomial, we express it as the product of its prime factors. For example, $x^3 - x = x(x^2 - 1)$ is not completely factored in the second form because $x^2 - 1$ is not prime.

2.2. *Useful identities*

Certain operations in algebra are repeated so often that much time is saved by memorizing them in type forms instead of working them out on each occasion. This is especially true of the following results, which may be considered as multiplications as they stand or as factorizations as read from right to left.

(1) $$\boldsymbol{a(b + c) = ab + ac.}$$

(2) $$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2.}$$

(3) $$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2.}$$

(4) $$\boldsymbol{(a - b)(a + b) = a^2 - b^2.}$$

The student should check these results by multiplication, and should verify them for various special values of the letters, such as $a = 4$, $b = 2$, etc. They are examples of *algebraic identities.*

An *identity* is an equation which is true for all permissible values of the letters involved. A value is permissible if it makes each side of the equation a definite number. In the identity

$$\frac{2x}{x^2 - 1} = \frac{1}{x - 1} + \frac{1}{x + 1}$$

the values 1 and -1 for x are not permissible. Why?

Also useful from the standpoint of factoring are the identities:

(5) $$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2).}$$

(6) $$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2).}$$

Illustrative Examples

A. Type Products

1. $3(2x^2 - 3x + 1) = 3(2x^2) - 3(3x) + 3 \cdot 1$
$= 6x^2 - 9x + 3.$

2. $(2x + 3y)^2 = (2x)^2 + 2(2x)(3y) + (3y)^2$
$= 4x^2 + 12xy + 9y^2.$

3. $(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4.$

4. $(5x - 4)(5x + 4) = (5x)^2 - 4^2 = 25x^2 - 16.$

B. Examples in Factoring

5. $2ax^2 - 6ax + 2a = 2a(x^2 - 3x + 1).$

6. $4x^2 - 9 = (2x)^2 - 3^2 = (2x - 3)(2x + 3).$

7. $4x^2 + 4ax + a^2 - y^2 + 2yb - b^2$
$= (4x^2 + 4ax + a^2) - (y^2 - 2yb + b^2)$
$= (2x + a)^2 - (y - b)^2$
$= [(2x + a) - (y - b)][(2x + a) + (y - b)]$
$= (2x + a - y + b)(2x + a + y - b).$

8. $8x^3 - 27 = (2x)^3 - 3^3$
$= (2x - 3)[(2x)^2 + (2x)(3) + 3^2]$
$= (2x - 3)(4x^2 + 6x + 9).$

9. $27y^3 + 1 = (3y)^3 + 1^3$
$= (3y + 1)[(3y)^2 - (3y)(1) + 1^2]$
$= (3y + 1)(9y^2 - 3y + 1).$

10. $9x^2 + 24xy + 16y^2 = (3x + 4y)^2.$

11. $4a^2 - 20a + 25 = (2a - 5)^2.$

EXERCISE 5

1. State each of the identities (1), (2), (3) and (4), Art. 2.2, in English.

2. Write a trinomial involving x which is (a), rational and integral in x, (b), not rational and integral in x.

By use of the proper type products perform (orally when possible) the multiplications indicated in problems 3–29.

3. $5x(2x - a - 1)$.
4. $-3y(y - 2xy)$.
5. $4a^2b(b^3 - ab)$.
6. $-2x^2(3x - 2ax + b + 2)$.
7. $6a(-2x - a)$.
8. $3p(p^2 + q^2)$.
9. $(x + 5)^2$.
10. $(2x^2 + 4)^2$.
11. $-(3xy + 2)^2$.
12. $-2(3m^2 + 2n)^2$.
13. $(2a^2 - b)^2$.
14. $(4m^2 - 3x)^2$.
15. $-2(-3y + x)^2$.
16. $(2 - y^2)^2$.
17. $(3a^2c - \frac{1}{2})^2$.
18. $(3x - 2y)(3x + 2y)$.
19. $(2x^2 + 4y^2)(2x^2 - 4y^2)$.
20. $-(2x^3 - y^3)(2x^3 + y^3)$.
21. $\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{a}{b} - \frac{b}{a}\right)$.
22. $[(x + a) - 2][(x + a) + 2]$.
23. $(3x - 2 - 3y)(3x - 2 + 3y)$.

HINT: $3x - 2 - 3y = [(3x - 2) - 3y]$.

24. $[(x + y) - (2r - s)][(x + y) + (2r - s)]$.
25. $(x + y)(x^2 - xy + y^2)$.
26. $(3a + b)(9a^2 - 3ab + b^2)$.
27. $(2p^2 + 3mp)(4p^4 - 6mp^3 + 9m^2p^2)$.
28. $-(y - a)(y^2 + ay + a^2)$.
29. $-3(2a - b)(4a^2 + 2ab + b^2)$.

30. Prove the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ both by direct multiplication and by writing the left member in the form $[(a + b) + c]^2$.

31. State the identity in problem 30 in a suitable form for oral problems.

Use the statement obtained in problem 31 *to find the squares indicated in problems* 32–34.

32. $(x + y + 2z)^2$.
33. $(3x - 2y + z)^2$.
34. $(2a - b - 3c)^2$.

Factor the expressions in problems 35–60.

35. $6ax^2 + 2ax - 4a$.
36. $x(a + b) + x$.
37. $2ay^2 - y$.
38. $4x^2 + 4x + 1$.

39. $x^2 - 6xy + 9y^2$.

40. $4x^2 - 9y^2$.

41. $4y^2 - 1$.

42. $(a + b)^2 - (c + d)^2$.

43. $1 - (2x - y)^2$.

44. $8 - 27y^3$.

45. $27x^3 + 1$.

46. $27x^6 - 8y^6$.

47. $(a + b)^3 - 8$.

48. $64 - (x + 2y)^2$.

49. $(x + y)^2 - (x + y)$.

50. $1 - 16y^4$.

51. $x^6 - y^6$.

52. $x^6 + y^6$.
HINT: $x^6 + y^6 = (x^2)^3 + (y^2)^3$.

53. $16a^2 - 25b^2$.

54. $(x - y)^2 - (a + b)^2$.

55. $(m - n)^2 - 16$.

56. $(k + 1)^2 - (x - y)^2$.

57. $100x^5y^3 - 25x^3y$.

58. $12a^4b^5 - 36$.

59. $64x^3y^5 - 16xy^3$.

60. $72a^4b^2 - 9ab^2$.

2.3. *Factoring a trinomial*

The identities

$$(x + a)(x + b) = x^2 + (a + b)x + ab, \tag{1}$$

and

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \tag{2}$$

are not worth memorizing in themselves; but they suggest a practical method of multiplying two monomials and thereby aid in the factoring of a trinomial. To illustrate, we may find the product $(2x + 3)(3x - 2)$ mentally by use of the following scheme. The first term of the product is $(2x)(3x) = 6x^2$, The second term is the sum of the products of "inside terms" and "outside terms," or $3(3x) + 2x(-2) = 9x - 4x = 5x$. The third term is $3(-2) = -6$. Hence the product is $6x^2 + 5x - 6$.

To factor the trinomial $6x^2 + 11x - 2$, for example, we experiment with binomial factors whose first terms are $6x$ and x or $3x$ and $2x$, and whose second terms are factors of -2. Among the possibilities are $(6x + 1)(x - 2)$, $(6x - 1)(x + 2)$, $(6x + 2)(x - 1)$, $(6x - 2)(x + 1)$, $(3x + 1)(2x - 2)$, etc. A quick inspection of "inside plus outside products" (the

key to the whole method) yields $x - 12x = -11x$, $-x + 12x = 11x$, and we stop right there, having found that $6x^2 + 11x - 2 = (6x - 1)(x + 2)$. The student should *always* check the middle term in the product of the trial factors before accepting them as correct.

The method is unchanged but shorter when the coefficient of x^2 is 1. In factoring $x^2 + x - 12$, for example, we seek two numbers whose product is -12 and whose sum is 1, the coefficient of x. These are readily found to be 4 and -3. We write the factors $(x + 4)(x - 3)$ and check rapidly the inside and outside products. It is more important for the student to become accustomed to this check as a *method* than it is for him to learn (2) or (1) as formulas.

EXERCISE 6

Do orally problems 77–96 in the Review Exercises of Chapter 1.

Factor the following trinomials.

1. $x^2 + 2x + 1$.	**2.** $4x^2 + 12x + 9$.	**3.** $16x^2 + 40x + 25$.
4. $a^2 - 10a + 25$.	**5.** $9k^2 + 6k + 1$.	**6.** $x^2 - 4x + 4$.
7. $9x^2 - 12x + 4$.	**8.** $25x^2 - 40x + 16$.	**9.** $4m^2 - 4m + 1$.
10. $36r^2 - 24rs + 4s^2$.	**11.** $x^2 - 5x + 6$.	**12.** $x^2 - 2x - 3$.
13. $x^2 + 3x + 2$.	**14.** $x^2 - 5x - 6$.	**15.** $x^2 + 6x - 7$.
16. $x^2 + 5x - 6$.	**17.** $x^2 - 3x + 2$.	**18.** $x^2 - 6x - 7$.
19. $x^2 + x - 2$.	**20.** $x^2 - x - 2$.	**21.** $x^2 - 11x + 30$.
22. $x^2 - 11x - 12$.	**23.** $x^2 + 11x - 12$.	**24.** $x^2 - 4x - 5$.
25. $x^2 + 4x - 5$.	**26.** $x^2 - 10x + 21$.	**27.** $x^2 + 12x + 32$.
28. $x^2 + 12x - 13$.	**29.** $x^2 - 12x - 13$.	**30.** $x^2 + 10x - 11$.
31. $2x^2 + 5x + 2$.	**32.** $2x^2 - 3x - 2$.	**33.** $3x^2 + 5x - 2$.
34. $5x^2 - 3x - 2$.	**35.** $4x^2 - x - 3$.	**36.** $6x^2 + 5x - 6$.
37. $6x^2 - 5x - 6$.	**38.** $10x^2 - 11x + 3$.	**39.** $10x^2 - x - 3$.
40. $8x^2 - 14x - 15$.	**41.** $6x^2 + 13x + 5$.	**42.** $15x^2 - 19x + 6$.
43. $9x^2 - 3x - 2$.	**44.** $9x^2 + 15x - 6$.	**45.** $14x^2 + 11x - 15$.
46. $24x^2 + 7x - 6$.	**47.** $27x^2 - 6x - 8$.	**48.** $20x^2 + 16x + 3$.
49. $36x^2 + 9x - 10$.	**50.** $27x^2 + 3x - 2$.	

2.4. *Factoring by grouping terms*

In general, factoring is more difficult than multiplication because there is no routine method which can always be applied. Yet there is one device which often proves useful — that of grouping terms.

Since $cx + dx = x(c + d)$, by the distributive law of multiplication, it follows that, if we replace x by $(a + b)$,

$$(1) \qquad c(a + b) + d(a + b) = (a + b)(c + d).$$

That is, if we can arrange the terms of an expression in groups connected by plus and minus signs in such a way that a certain quantity appears as a factor of each group, then that quantity is a factor of the whole expression.

Certain frequently made errors will be avoided if the student will *study carefully* the examples below.

Illustrative Examples

1. $2ax + 3ay + 4bx + 6by$
$= (2ax + 3ay) + (4bx + 6by)$
$= a(2x + 3y) + 2b(2x + 3y)$
$= (2x + 3y)(a + 2b).$

2. $12ac - 8bc - 3ad + 2bd$
$= (12ac - 8bc) - (3ad - 2bd)$
$= 4c(3a - 2b) - d(3a - 2b)$
$= (3a - 2b)(4c - d).$

3. $3ax + x - y + 3az + z - 3ay$
$= 3ax - 3ay + 3az + x - y + z$
$= (3ax - 3ay + 3az) + (x - y + z)$
$= 3a(x - y + z) + 1(x - y + z)$
$= (x - y + z)(3a + 1).$

If no common factor appears after various experimental groupings of the terms, it may be possible to arrange them so that the expression to be factored is seen to be the difference of two squares.

Illustrative Examples

4. $4x^2 - 9y^2 + 6y - 1$
$$= 4x^2 - (9y^2 - 6y + 1)$$
$$= (2x)^2 - (3y - 1)^2$$
$$= [(2x) - (3y - 1)][(2x) + (3y - 1)]$$
$$= (2x - 3y + 1)(2x + 3y - 1).$$

5. $4xy - 1 + 4y^2 + x^2$
$$= (x^2 + 4xy + 4y^2) - 1$$
$$= (x + 2y)^2 - 1^2$$
$$= [(x + 2y) - 1][(x + 2y) + 1]$$
$$= (x + 2y - 1)(x + 2y + 1).$$

It is worth noting that a definite procedure is indicated for factoring, or attempting to factor, a polynomial of four terms. The "two-two" grouping will aid in exposing any binomial factor. If this fails and there is at least one perfect square term in the expression, the "three-one" grouping may be tried.

A binomial or trinomial containing two perfect square terms can sometimes be changed to a perfect square by the addition of a term. If the term thus added is a perfect square itself, the original expression may be factored, as in the examples below.

Illustrative Examples

6. $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2$
$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x).$$

7. $x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2 = (x^2 + 1)^2 - x^2$
$$= (x^2 + 1 - x)(x^2 + 1 + x).$$

EXERCISE 7

Factor the following expressions.

1. $a(x - y) + b(x - y)$.
2. $b(x + y) - c(x + y)$.
3. $x(a + b) + y(a + b)$.
4. $m(k - 1) - n(k - 1)$.
5. $-h(r + s) + k(r + s)$.
6. $ax + ay + bx + by$.

7. $bx - by + cx - cy$.
8. $x^3 - 2x^2 + x - 2$.
9. $2x^3 + 3x^2 - 2x - 3$.
10. $ax^2 + ax - bx - b$.
11. $x^2 - y^2 + x - y$.
12. $x^2 - y^2 - x + y$.
13. $4x^2 - 9y^2 + 2x - 3y$.
14. $9x^2 - 16y^2 - 3x - 4y$.
15. $16x^2 - 25y^2 + 8ax - 10ay$.
16. $4x^2 + 2y - 1 - y^2$.
17. $x^2 + y^2 + 2xy - 9$.
18. $16 + 2xy - x^2 - y^2$.
19. $x^2 + y^2 - 2xy - 25$.
20. $16 - 8x + x^2 - y^2$.
21. $y^2 - x^2 + 2y + 1$.
22. $4x^2 - y^2 - 4x + 1$.
23. $x^2 - y^2 - 6x + 9$.
24. $x^2+a^2-b^2-2ax+2b-1$.
25. $x^2 - y^2 - c^2 - 4x + 2cy + 4$.
26. $x^8 + x^4 + 1$.
27. $x^4 - 3x^2 + 1$.
28. $x^4 + 3x^2 + 4$.
29. $x^4 + 5x^2 + 9$.
30. $x^4 - 5x^2 + 4$.
31. $x^4 - 7x^2 + 9$.
32. $x^4 + 2x^2 + 9$.
33. $x^4 + 4x^2 + 16$.
34. $4x^4 + 3x^2 + 1$.
35. $ax + by + bx + ay$.
36. $x + y - x^2 + y^2$.
37. $4x^2 - y^2 + 4y - 4$.
38. $4x^4 + 1$.
39. $25x^4 + x^2 + 1$.
40. $9x^4 + 2x^2 + 1$.
41. $9x^4 - 10x^2 + 1$.
42. $9x^4 - 40x^2 + 16$.

2.5. *Definitions*

The *degree* of an integral rational term in any letters is the sum of the exponents of the letters in that term. Its degree in a given letter is the exponent of that letter. Thus $3x^2y^3$ is of degree 5 in x and y, of degree 2 in x, and of degree 3 in y.

The degree of a polynomial is the same as that of its term of highest degree. For example, $x^3y + xy^5 - 1$ is a sixth degree polynomial of degree 3 in x and of degree 5 in y.

A polynomial A is a *multiple* of a polynomial B if B is a factor of A. To illustrate, 5, 10, $15x$ and $25(x + y)$ are multiples of 5, while $x - 1$ and $x^2 - 1$ are multiples of $x - 1$.

A multiple of each of two or more polynomials is called a *common multiple*. For instance, $25x^3y^2$ is a common multiple of 5, x and y^2.

2.6. *Lowest common multiple*

The lowest common multiple (L.C.M.) of two or more polynomials is the common multiple of lowest degree and with the smallest possible numerical coefficients. We find it by multiplying the prime factors of all the polynomials, assigning to each the largest exponent which it has in any one polynomial. Factors such as $x - y$ and $y - x$, which differ only in sign, should be considered as the same.

Example 1. The L.C.M. of 6, 9, and 12, or of $2 \cdot 3$, $3 \cdot 3$, and $2^2 \cdot 3$, is $2^2 \cdot 3^2$, or 36.

Example 2. The L.C.M. of $6x$, $9(y + 1)^3$, and $12x^2(y + 1)$ is $36x^2(y + 1)^3$.

Example 3. The L.C.M. of $x - y$ and $y^2 - x^2$ is $y^2 - x^2$ *or* $x^2 - y^2$, and not $(x - y)(y^2 - x^2)$. (Note that there are two correct forms of the L.C.M. one being the negative of the other.)

2.7. *Highest common factor*

The highest common factor (H.C.F.) of two or more polynomials is the common factor of highest degree and with the largest possible numerical coefficients. We find it by multiplying the common factors, assigning to each the smallest exponent which it has in any one polynomial.

Example 1. The H.C.F. of 30, 12, and 18, or of $2 \cdot 3 \cdot 5$, $2^2 \cdot 3$, and $2 \cdot 3^2$, is $2 \cdot 3$ or 6.

Example 2. The H.C.F. of $30x^3(x - y)^4$, $12x^2(x - y)$, and $18x^4(y - x)^3$ is $6x^2(x - y)$, or $6x^2(y - x)$.

EXERCISE 8

Find the H.C.F. and L.C.M. of the polynomials in each group.

1. 9, 12, 15. **2.** 12, 18, 24. **3.** 18, 27, 36.

4. 14, 21, 36. **5.** 20, 35, 50. **6.** 30, 45, 75.

7. 80, 32, 96. **8.** 54, 36, 90. **9.** 65, 91, 104.

10. $x^2, 3x^3y, 4x^4y^3$.

11. $5a^3x^2, 15a^2x^3, 10a^4x^4$.

12. $12x^3y^2, 15x^2y^3, 9x^4y^4$.

13. $(x - y), (x^2 - y^2), (x^2 - xy)$.

14. $(x^2 + xy), (x + y), (x^2 - y^2)$.

15. $(x^2 - 2xy + y^2), (x^2 - y^2), (x^3 - x^2y)$.

16. $(x^2 + 2xy + y^2), (x^2 - y^2), (x^3 + x^2y)$.

17. $(x^2 - 3x + 2), (x^2 - 4), (x^2 - 5x + 6)$.

18. $(2x^2 - 3x + 1), (2x^2 + 5x - 3), (4x^2 - 1)$.

19. $(3x^2 + 5x - 2), (3x^2 - 4x + 1), (3x^2 + 2x - 1)$.

20. $(2x^2 - 5x - 3), (2x^2 - x - 1), (2x^2 + 3x + 1)$.

21. $(6x^2 + x - 1), (3x^2 - 4x + 1), (6x^2 - 5x + 1)$.

22. $(5x^2 - x - 4), (15x^2 + 2x - 8), (10x^2 - 7x - 12)$.

23. $(x - 2y)^2, (2y - x)^3, (x^3 - 2x^2y)$.

24. $(x - y)^3, (y - x)^3, (xy^2 - y^3)^2$.

REVIEW EXERCISES

Factor by using the type form $ab + ac = a(b + c)$.

1. $4y^3 - 12y^2$.

2. $3ab^2 + 6b^3$.

3. $m^2n - mn^2$.

4. $3a^3b + 5ab^2 - ab$.

5. $6x^5 - 2x^3y^3 - 2x^3y$.

6. $y(x - 3y) + 2(x - 3y)$.

7. $c(3a - b) - 2a(3a - b)$.

8. $-x^2(2m - n) - y^2(2m - n) + 3(2m - n)$.

Factor by grouping.

9. $mx + my + cx + cy$.

10. $a^4 + 2a^3 - 2a - 4$.

11. $x^3 - x^2 - x + 1$.

12. $6ab - 2a + 3b - 1$.

13. $x^3 - 4x^2y - 2xy^2 + 8y^3$.

14. $6pq - 9p + 4q - 6$.

15. $6m^3 + 12m^2n + 4mn^2 + 8n^3$

16. $x^2 - 9y^2 + x - 3y$.

Factor the following trinomials.

17. $y^2 - y - 12$.

18. $2x^2 + 5x + 2$.

19. $x^2 + 10x + 21$.

20. $m^2n^2 - 5mn - 15$.

21. $x^2 + xy - 12y^2$.

22. $p^4 + 16p^2 - 36$.

23. $12a^2 - a - 1$.

24. $25r^2 + 10rs - 3s^2$.

25. $a^2 - 5ab - 24b^2$.

26. $12x^2 + 71x - 25$.

27. $20 + 8x - x^2$.

28. $x^8 + 4x^4y^4 + 4y^8$.

29. $4c^2 - 4cd + d^2$.

30. $9a^2 + 6a + 1$.

31. $4 - 20x + 25x^2$.

32. $9x^4 - 6x^2yz + y^2z^2$.

Factor by using the type form $a^2 - b^2 = (a - b)(a + b)$.

33. $4x^2 - 121y^2$.

34. $81 - x^4$ (three factors).

35. $16a^2 - 25b^2$.

36. $(3x - y)^2 - 9$.

37. $x^2 - (2y + z)^2$.

38. $(x + 2y)^2 - 9(m - n)^2$.

39. $1 - (x + y)^2$.

40. $4x^2 - 4xy + y^2 - z^2$.

HINT. $4x^2 - 4xy + y^2 - z^2 = (2x - y)^2 - z^2$.

41. $9 - a^2 + 8ab - 16b^2$.

42. $4x^2 - 4ax + 2by + a^2 - b^2 - y^2$.

43. $p^2 + q^2 - a^2 + 2pq$.

Factor by using the type forms $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$; and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

44. $x^3 + 8$.

45. $m^3 - 27$.

46. $a^3y^3 + 64$.

47. $x^6 + a^6$.

48. $(2x - 1)^3 - 8$.

49. $27y^3 + (a + 2b)^3$.

50. $x^6 - y^6$.

HINT. *First treat as difference of squares.*

Factor as a difference of two squares by adding and subtracting a perfect square.

51. $x^4 + 64$.

52. $x^4 + 4y^4$.

53. $x^4 + x^2y^2 + y^4$.

54. $x^4 - 24x^2 + 16$. HINT. $x^4 - 24x^2 + 16 = (x^2 - 4)^2 - 16x^2$.

55. $x^4 - 12x^2y^2 + 16y^4$.

56. $a^4 - 10a^2b^2 + 16b^4$.

Factor completely, using the type forms given in this chapter. Remove first any monomial divisor which exists in the given expression before factoring it further.

57. $4r^2y^2 - 25r^2$.

58. $y^6 - 8y^4 + 16y^2$.

59. $x^4 - a^4$.

60. $2a^6 - 8y^6$.

61. $a^3 + a^2 - 4a - 4$.

62. $y^4 - 2a^2y^2 + a^4$.

63. $m(n^2 - 1) - 2(n^2 - 1)$.

64. $x^3 + 27m^3$.

65. $4x^2 - 1 - 4y - 4y^2$.

66. $(2x - 4)^2 - (3y + 2)^2$.

67. $3x^2a - 2axy - ay^2$.

68. $8r^2mx - 16rm - 48r^3m^3$.

69. $(x + y) - b^2(x + y)$.

70. $2(a + b)^2 + 5(a + b) - 12$.

71. $p^2q^2 - 1$.

72. $4x^2 + 28cx + 49c^2$.

73. $x^3 - (a + 2)^3$.

74. $x^2(2x + 1) - (4x^2 - 1)$.

75. $(a + b)(x + y) - (a + b)(3x - y)$.

76. Why is it that $\frac{10^3 + 7^3}{10^3 + 3^3}$ can be simplified by the illegitimate operation of canceling the exponents?

77. Show that $\frac{(a + b)^3 + a^3}{(a + b)^3 + b^3} = \frac{2a + b}{a + 2b}$, and apply this result to to problem 76.

Chapter Three

FRACTIONS

3.1. *Definitions*

An ***algebraic fraction*** *is an indicated quotient of two algebraic expressions.* The dividend and divisor are called respectively the *numerator* and *denominator.*

Examples. $\dfrac{x}{y}$; $\dfrac{a+1}{\dfrac{a}{2}-1}$; $\dfrac{2-3x}{1}$; $\dfrac{2}{3}$; $\dfrac{\dfrac{x}{a}+\dfrac{y}{2}}{x-\dfrac{y}{3}}$.

A fraction is *simple* if neither its numerator nor its denominator contains a fraction; otherwise it is *complex.* For example, $\dfrac{2}{3}$ and $\dfrac{2x+1}{2x-1}$ are simple, while $\dfrac{2+\dfrac{1}{2}}{3}$ and $\dfrac{2+\dfrac{1}{x}}{\dfrac{3}{x}-1}$ are complex.

If the product of two numbers is 1, each is called the *reciprocal* of the other. Thus the reciprocal of $\dfrac{a}{b}$ is $\dfrac{b}{a}$; and the reciprocal of $-x$ $\left(\text{or } \dfrac{-x}{1}\right)$ is $\dfrac{1}{-x}$, which equals $-\dfrac{1}{x}$. Zero is the only number which has no reciprocal. Why?

3.2. *Reduction of a simple fraction to its lowest terms*

LAW 1. *The value of a fraction is not changed when the numerator and denominator are multiplied or divided by the same quantity (except zero).*

Examples. $\dfrac{3}{6} = \dfrac{1}{2}$; $\dfrac{2}{3} = \dfrac{8}{12}$; $\dfrac{ax}{xy} = \dfrac{a}{y}$; $\dfrac{a}{b} = \dfrac{x^2a}{x^2b}$.

By virtue of this law we may *reduce to lowest terms* a simple fraction by factoring the numerator and denominator and striking out the common factors. This operation, called *canceling*, is a short way of dividing both numerator and denominator by a common factor.

$$\textit{Example.}\ \frac{ax^2 - a}{ax^3 - a} = \frac{\overset{1}{\cancel{a}}\overset{1}{\cancel{(x - 1)}}(x + 1)}{\underset{1}{\cancel{a}}\underset{1}{\cancel{(x - 1)}}(x^2 + x + 1)} = \frac{x + 1}{x^2 + x + 1}.$$

A fraction is not reduced to lowest terms until all common factors of the numerator and the denominator have been canceled; or in other words, until the numerator and denominator have been divided by their highest common factor.

When all of the numerator or denominator has been canceled, the factor 1 always remains.

$$\textit{Example.}\ \frac{\overset{1}{\cancel{a}}}{\underset{1}{\cancel{a}}b} = \frac{1}{b};\quad \frac{\overset{1}{\cancel{a}}b}{\underset{1}{\cancel{a}}} = \frac{b}{1} = b;\quad \frac{\overset{1}{\cancel{ab}}}{\underset{1}{\cancel{ab}}} = \frac{1}{1} = 1.$$

CAUTION. Errors due to faulty cancellation are *very* numerous, and the student who is not sure of his knowledge or ability in algebra should be especially careful when he is tempted to scratch out letters or expressions just because they are above or below a line. The simple rule to be applied is that *anything canceled must be a factor of the whole numerator and also a factor of the whole denominator.* Hence the student should make sure that this factoring is done correctly *before* he does any canceling at all. For example, if the numerator is $xy + a$, then neither x nor y nor a is a factor of the whole numerator, although the expression *as a whole* is a factor of itself, as shown in some of the examples.

Examples in which canceling is permissible.

$$\frac{xy+a}{b(xy+a)} = \frac{\overset{1}{\cancel{(xy+a)}}}{b\underset{1}{\cancel{(xy+a)}}} = \frac{1}{b}; \quad \frac{2x+4}{6} = \frac{\overset{1}{\cancel{2}}(x+2)}{\underset{1}{\cancel{2}}\cdot 3} = \frac{x+2}{3}.$$

Examples in which canceling is incorrect. (Decide why in each case.)

$$\frac{xy+a}{x}; \frac{a+1}{a+2}; \frac{2(x+y)}{1-3(x+y)}; \frac{2x+y+z}{x+y+z}; \frac{(a+b)(c+d)}{(a+b)c+d}.$$

In the fraction $\frac{a+1}{a+2}$ the canceling of a is of course not permissible since it would violate the rule about factors. But *why* is it not permissible? Note that when we cancel a factor we are actually *dividing* the numerator and denominator by that factor, whereas if we incorrectly strike out a from the fraction $\frac{a+1}{a+2}$ we are *subtracting* a from the numerator and denominator. This changes the value of a fraction, as illustrated in the case $\frac{2}{3} \neq \frac{2-1}{3-1}\left(\text{or } \frac{2}{3} \neq \frac{1}{2}\right)$.

It is permissible to cancel $x + 2$ from the numerator and denominator of $\frac{3y\overset{1}{\cancel{(x+2)}} - 5\overset{1}{\cancel{(x+2)}}}{\underset{1}{\cancel{(x+2)}}(x-2)}$ obtaining $\frac{3y-5}{x-2}$, since this amounts to dividing both members of the fraction by the canceled factor. However, it is a safer procedure to write this fraction as $\frac{(x+2)(3y-5)}{(x+2)(x-2)}$ before carrying out cancellation. It will help to avoid errors if both the full numerator and the full denominator of a fraction are completely factored before cancellation.

EXERCISE 9

1. How many *terms*, and how many prime *factors*, has each of the following expressions: (a) xyz; (b) $x^2 + x + 1$; (c) $x^2 - y^2$?

2. In which of the two following cases is canceling always permissible: The numerator and denominator of a fraction have (a) a common term, (b) a common factor?

In which of problems 3–13 *is canceling permissible? In these cases reduce the fractions to lowest terms.*

3. $\dfrac{ax}{x(b+1)}$.

4. $\dfrac{x+y}{x-y}$.

5. $\dfrac{(x-y)(x+y)+1}{2(x+y)}$.

6. $\dfrac{(x-1)(x+1)}{3(x+1)}$.

7. $\dfrac{(x-1)x+1}{x+1}$.

8. $\dfrac{(x+1)a}{x+1}$.

9. $\dfrac{(x+1)a}{(x+1)+b}$.

10. $\dfrac{(x-y)(x+y)}{x-y(x+y)}$.

11. $\dfrac{a+b+c}{3(x+y)(a+b+c)}$.

12. $\dfrac{(a+b)x-y}{(a+b)(x-y)}$.

13. $\dfrac{(x-y)(x+y)}{ax+ay}$.

Explain the errors which have been made in cancellations for problems 14–17.

14. $$\frac{3(x+y) + \overset{1}{\cancel{(x-y)}}}{\underset{1}{\cancel{(x-y)}}(x+y)} \overset{?}{=} \frac{3x+3y+1}{x+y}.$$

15. $$\frac{x + \overset{1}{\cancel{y-z}}}{\underset{1}{\cancel{y-z}}} \overset{?}{=} x + 1.$$

16. $$\frac{\overset{1}{\cancel{x+y}}}{\underset{x+y}{\cancel{x^2-y^2}} + 3} \overset{?}{=} \frac{1}{x+y+3}.$$

17. $\dfrac{\overset{(a+b)}{\cancel{(a+b)^2}} + \overset{1}{\cancel{c^2}}}{\underset{1}{\cancel{c^2}}\underset{1}{\cancel{(a+b)}}} \stackrel{?}{=} a + b + 1.$

Reduce to lowest terms.

18. $\dfrac{2}{6}$. 19. $\dfrac{66}{99}$. 20. $\dfrac{15}{35}$. 21. $\dfrac{42}{30}$. 22. $\dfrac{x^3 - 1}{x^2 - 1}$.

23. $\dfrac{36x^2}{54x}$. 24. $\dfrac{x(y - 3)}{2(y - 3)}$. 25. $\dfrac{7(x - 5)}{3(x - 5)}$.

26. $\dfrac{6(x^2 - y^2)}{5(x - y)}$. 27. $\dfrac{5x(y^2 - 1)}{2x(y - 1)}$. 28. $\dfrac{7y^2(x^2 - 9)}{3y(x - 3)}$.

29. $\dfrac{x^2 - 3x - 4}{x^2 + 5x + 4}$. 30. $\dfrac{x^2 - 3x + 2}{x^2 - x - 2}$. 31. $\dfrac{x^2 + 7x + 12}{x^2 + x - 12}$.

32. $\dfrac{x^2 - 2x - 15}{x^2 - 9}$. 33. $\dfrac{x^2 - 5x + 6}{x^2 - x - 6}$. 34. $\dfrac{x^2 + x - 6}{x^2 + 2x - 3}$.

35. $\dfrac{2x^2 + 3x - 2}{2x^2 - 3x + 1}$. 36. $\dfrac{6x^2 - x - 2}{2x^2 - x - 1}$. 37. $\dfrac{3x^2 + 2x - 1}{2x^2 + x - 1}$.

38. $\dfrac{12x^2 + 5x - 3}{8x^2 + 10x + 3}$. 39. $\dfrac{15x^2 - 2x - 8}{-12 + 7x + 10x^2}$.

40. $\dfrac{21x^2 + 43x - 14}{-21 + 5x + 6x^2}$. 41. $\dfrac{x^3 - x^2 - 4x + 4}{x^3 - x^2 - 9x + 9}$.

42. $\dfrac{x^3 - 3x^2 + x - 3}{-3x + x^2 + 3x^3 - x^4}$. 43. $\dfrac{x^3 - 5x^2 + 2x - 10}{-15 + 3x + 5x^2 - x^3}$.

44. $\dfrac{5(c - d)^3 - x(c - d)}{c^2 - d^2}$. 45. $\dfrac{r^2 - s^2 - 5r + 5s}{r^2 - 2rs + s^2}$.

46. $\dfrac{x^4 + 4y^4}{5x^2 - 10xy + 10y^2}$. 47. $\dfrac{(x + y)^2 + 11(x + y) + 24}{x^2 + 2xy + y^2 - 9}$.

48. $\dfrac{(x + y)(x - y) - 5y(x - y)}{x^3 - y^3}$.

49. $\dfrac{a^3 + 8b^3 + a^2 - 4b^2}{a^2 - 4b^2}$.

50. $\dfrac{(3x - 2y)(3x^2 + xy - 4y^2)}{(3x + 2y)(3x + xy - 2y^2)}$.

3.3. *Rules about fractions*

In this and the next two articles we shall deal with simple fractions, such as $\frac{2}{3}$ or $\frac{3x - 1}{x + 2}$, whose numerators and denominators are polynomials.

RULE 1. *The sign before a fraction is changed when the sign of either the numerator or the denominator is changed.*

For by the rule of signs [(4), Art. 1.5], $\frac{a}{-b} = -\frac{a}{b}$, and $\frac{-a}{b} = -\frac{a}{b}$.

RULE 2. *The sign before a fraction is changed when the sign of a factor of either the numerator or the denominator is changed.*

For again by the rule of signs, $a(-b) = -ab$, and hence when the factor b is changed to $-b$ the sign of the whole numerator or denominator in which it appears will be changed, so that Rule 1 will apply.

Example 1. $\frac{1}{(-2)(-3)} = -\frac{1}{2(-3)} = \frac{1}{2 \cdot 3} = \frac{1}{6}$.

Example 2. $\frac{a}{x(1 - x)} = -\frac{a}{x(-1 + x)}$, or $-\frac{a}{x(x - 1)}$.

Reminder. It is important to note here that Rule 2, as does Rule 3 below, refers to *factors* and not *terms*. For instance, in Example 2 above, when the sign of the factor $1 - x$ is changed it becomes $-1 + x$ or $x - 1$, and *not* $1 + x$.

Definition. Integers divisible by 2 are *even;* all other integers are *odd.* Even numbers include 2, 4, 6, -2, -4, etc., as well as 0; odd numbers include 1, 3, 5, -1, -3, etc.

RULE 3. *The sign before a fraction is or is not changed according as the number of factors whose signs are changed is odd or even.* The altered factors may be all in the numerator or in the denominator, or they may be in both.

Example.

$$\frac{(1-x)(2-x)(x-3)}{(x-1)(x-2)(3-x)} = -\frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x-3)} = -1.$$

Here the minus sign is inserted before the fraction in the second step because the number of factors there changed in sign is 3, which is odd.

RULE 4. *To add or subtract fractions with the same denominators, add or subtract the numerators and place the result over the common denominator.*

Examples. $\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$; $\frac{8}{5} - \frac{6}{5} = \frac{8-6}{5} = \frac{2}{5}$;

$$\frac{6}{7} + \frac{5}{7} - \frac{8}{7} = \frac{6+5-8}{7} = \frac{3}{7};$$

$$\frac{x}{a+1} + \frac{x^2}{a+1} - \frac{2}{a+1} = \frac{x+x^2-2}{a+1};$$

$$\frac{1}{x-1} + \frac{a}{1-x} = \frac{1}{x-1} - \frac{a}{x-1}$$

$$\text{(by Rule 1)} = \frac{1-a}{x-1}.$$

EXERCISE 10

In problems 1–9, *write each factor with a positive literal term, changing its sign where necessary.* [*For instance,* $x - 3$ *should be left unchanged, but* $3 - x$ *should be changed to* $-(x - 3)$.] *Then cancel factors where permissible and reduce each fraction to lowest terms.*

1. $\frac{x-1}{1-x}$.
2. $\frac{(x-1)^2}{(1-x)^2}$.
3. $\frac{(x-1)^3}{(1-x)^3}$.
4. $\frac{(x-1)(x-2)}{(1-x)(2-x)}$.
5. $\frac{(x-1)(2-x)(x-3)}{(1-x)(x-2)(3-x)}$.
6. $\frac{(3x-1)(4-x^3)(x-2)}{(2-x)(x^3-4)(3x-1)}$.
7. $\frac{(x-5)^2(x-6)^3}{(5-x)^2(6-x)^3}$.
8. $\frac{(x-1)10}{(1-x)10}$.
9. $\frac{(2x-3)^5(4-x)^6}{(3-2x)^5(x-4)^6}$.

Perform the operations indicated below.

10. $\dfrac{3x-1}{5}+\dfrac{x+2}{5}.$

11. $\dfrac{2x-2}{4}+\dfrac{2x+3}{4}.$

12. $\dfrac{3x+5}{6}-\dfrac{2x+3}{6}.$

13. $\dfrac{5x-7}{5}+\dfrac{3x+2}{5}.$

14. $\dfrac{7x+2}{8}-\dfrac{3x+3}{8}.$

15. $\dfrac{6x-5}{7}-\dfrac{5x-6}{7}.$

16. $\dfrac{x-3}{x+1}+\dfrac{2x+4}{x+1}.$

17. $\dfrac{3x+7}{2x-1}-\dfrac{2x+6}{2x-1}.$

18. $\dfrac{4x+5}{x-9}-\dfrac{3x-2}{9-x}.$

19. $\dfrac{3x-5}{1+x}+\dfrac{2x-3}{x+1}.$

20. $\dfrac{10x+3}{5x^2-6}+\dfrac{3+10x}{6-5x^2}.$

21. $\dfrac{15x-7}{3x^2-5}-\dfrac{7-15x}{5-3x^2}.$

3.4. *Addition and subtraction of simple fractions*

By use of Law 1, Art. 3.2, two or more simple fractions may be written with a common denominator.

Example 1. Write the numbers 4 (or $\frac{4}{1}$), $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{16}{15}$ as fractions with a lowest common denominator (L.C.D.).

Solution. The L.C.D. of the fractions is the L.C.M. (30) of the denominators 1, 3, 6, and 15. Now, $\dfrac{4}{1}=\dfrac{4\cdot 30}{1\cdot 30}=\dfrac{120}{30}$; $\dfrac{2}{3}=\dfrac{2\cdot 10}{3\cdot 10}=\dfrac{20}{30}$; $\dfrac{5}{6}=\dfrac{5\cdot 5}{6\cdot 5}=\dfrac{25}{30}$; and $\dfrac{16}{15}=\dfrac{16\cdot 2}{15\cdot 2}=\dfrac{32}{30}$.

Example 2. Write the fractions $\dfrac{a}{3x(x-2)}$ and $\dfrac{5}{6(2-x)}$ with a L.C.D.

Solution. $\dfrac{5}{6(2-x)}=-\dfrac{5}{6(x-2)}$ by Rule 1, Art. 3.3. The L.C.D. of the fractions is then $6x(x-2)$. We now write

$$\frac{a}{3x(x-2)}=\frac{2a}{6x(x-2)} \text{ and } \frac{-5}{6(x-2)}=-\frac{5x}{6x(x-2)}.$$

In view of Rule 4, Art. 3.3, we have

RULE 1. *To add or subtract fractions, write them with a common denominator (their L.C.D.), add or subtract the numerators as indicated, and put the result over the L.C.D.*

Example 1.

$$2 + \frac{1}{2} - \frac{5}{6} = \frac{2}{1} + \frac{1}{2} - \frac{5}{6} = \frac{12}{6} + \frac{3}{6} - \frac{5}{6}$$
$$= \frac{12 + 3 - 5}{6} = \frac{10}{6} = \frac{5}{3}.$$

Example 2.

$$\frac{x+2}{x+1} + \frac{2+x^2}{1-x^2} + x = \frac{x+2}{x+1} - \frac{x^2+2}{x^2-1} + \frac{x}{1}$$
$$= \frac{(x+2)(x-1)}{(x+1)(x-1)} - \frac{x^2+2}{x^2-1} + \frac{x(x^2-1)}{x^2-1}$$
$$= \frac{(x+2)(x-1)-(x^2+2)+x(x^2-1)}{x^2-1}$$
$$= \frac{x^2+x-2-x^2-2+x^3-x}{x^2-1}$$
$$= \frac{x^3-4}{x^2-1}.$$

From Rule 1 we get the simple results: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. But note that these results are not efficient as *formulas* when b and d have some factors in common. For example, if we use the method indicated by these results we have:

$$\frac{1}{x+1} + \frac{1}{x^2-1} = \frac{(x^2-1)+(x+1)}{(x+1)(x^2-1)} = \frac{x^2+x}{(x+1)(x^2-1)}$$
$$= \frac{x(x+1)}{(x+1)(x^2-1)} = \frac{x}{x^2-1},$$

whereas the method of Rule 1 gives the simplified answer at once.

Exercise. Find the blunder in the following incorrect addition: $\frac{1}{1} - \frac{x+2}{3} = \frac{3}{3} - \frac{x+2}{3} = \frac{3-x+2}{3} = \frac{5-x}{3}$. This typical error is so common that it should be thoroughly understood and then carefully avoided. A second common blunder is the omission of the denominator. The student should cultivate the habit of writing the denominator *first* to avoid forgetting it when the computation of the numerator is long.

EXERCISE 11

In each problem below, combine the given quantities into a single fraction and reduce it to lowest terms.

1. $\frac{1}{2} + \frac{1}{3}$.
2. $\frac{1}{3} + \frac{3}{5}$.
3. $\frac{3}{4} + \frac{1}{5}$.
4. $\frac{2}{7} + \frac{2}{3}$.
5. $\frac{3}{2} - \frac{5}{7}$.
6. $\frac{3}{5} + \frac{2}{3} - \frac{1}{4}$.
7. $\frac{3}{8} - \frac{4}{7} + \frac{1}{5}$.
8. $\frac{7}{5} - \frac{3}{4} + \frac{3}{8}$.
9. $\frac{8}{9} - \frac{1}{4} + \frac{2}{3}$.
10. $\frac{x+1}{1-x} + \frac{3x}{x-1}$.
11. $\frac{x+3}{x-2} + \frac{2x+3}{2-x}$.
12. $\frac{3x-5}{2x-3} - \frac{x-2}{3-2x}$.
13. $\frac{x}{2} + \frac{2x}{3} - \frac{1}{4}$.
14. $\frac{3x}{5} - \frac{x}{4} + \frac{2}{3}$.
15. $\frac{2x}{7} + \frac{x}{5} - \frac{3}{7}$.
16. $\frac{4x}{9} + \frac{2x}{3} - \frac{5}{6}$.
17. $\frac{5x}{4} - \frac{4x}{5} - \frac{3}{2}$.
18. $\frac{7x}{5} - \frac{3x}{2} + \frac{3}{5}$.
19. $\frac{2}{x} - \frac{x}{3} - \frac{5}{6}$.
20. $\frac{x}{5} + \frac{3}{x} - 3$.
21. $\frac{3}{2x} - \frac{x}{3} + 1$.
22. $\frac{3x}{5} - \frac{3}{5x} + 3$.
23. $\frac{7}{4x} - 5 + \frac{3x}{2}$.
24. $\frac{6x}{7} - 4 + \frac{3}{7x}$.
25. $\frac{3x-1}{x+2} + \frac{x-3}{x-2}$.
26. $\frac{3x+2}{x-3} + \frac{2x-3}{x+3}$.
27. $\frac{5x+6}{2x-1} + \frac{3x+2}{3x+2}$.
28. $\frac{3x+5}{2x+3} - \frac{2x+3}{3x-2}$.
29. $\frac{4x-5}{3x-4} - \frac{2x-7}{2x+3}$.
30. $\frac{9x-4}{4x+2} + \frac{3x+5}{x-3}$.

31. $\dfrac{7x+3}{2x-1}+\dfrac{x-2}{4x^2-1}.$

32. $\dfrac{5x-9}{x+3}-\dfrac{4x-3}{x+1}.$

33. $\dfrac{6x^2+7}{3x^2-2x-1}+\dfrac{2x-5}{3x+1}.$

34. $\dfrac{8x^2-5x+3}{6x^2+5x-6}-\dfrac{4x+7}{3x-2}+\dfrac{x+5}{2x+3}.$

35. $\dfrac{4x+3}{2x-5}+\dfrac{2x-5}{3x+4}-\dfrac{9x^2+6x+10}{6x^2-7x-20}.$

36. $\dfrac{10x-11}{3x-1}-\dfrac{5x+9}{2x+5}+\dfrac{x^2-3x+5}{6x^2+13x-5}.$

37. $\dfrac{2x-y}{3x-2y}-\dfrac{x-2y}{2y-3x}+2.$

38. $\dfrac{4x+7y}{5x-7y}-\dfrac{3x+4y}{7y-5x}-3.$

39. $\dfrac{4x-5y}{3x-y}+3+\dfrac{x+7y}{y-2x}.$

40. $\dfrac{7x+2y}{3x-5y}+\dfrac{2x-3y}{5y-3x}-2.$

41. $\dfrac{5x-3y}{2y-3x}-\dfrac{3x+5y}{3x-2y}.$

42. $\dfrac{9x-4y}{7x-y}-5+\dfrac{7x-5y}{y-7x}.$

43. $\dfrac{5}{x+y}-\dfrac{2x-3y}{x^2-xy+y^2}+1.$

44. $\dfrac{7}{x-y}-\dfrac{4x+2y}{x^2+xy+y^2}+1.$

45. $\dfrac{x+(2x-3)}{x}-\dfrac{3x+5}{2x-7}.$

46. $\dfrac{2x-(x-3)}{3x}-\dfrac{5x-6}{5x}.$

47. $\dfrac{(2x-3)-4x}{5x}+\dfrac{3x-2}{3x}.$

48. $\dfrac{(x+5)-3x}{3x}-\dfrac{2-3x}{2}.$

49. $\dfrac{(6x-5)-4x}{4x}-\dfrac{2x-5}{3}.$

3.5. *Multiplication and division of simple fractions*

RULE 1. *The product of two or more simple fractions is the product of the numerators divided by the product of the denominators.*

Example 1. $(2)\left(\dfrac{5}{3}\right)\left(\dfrac{1}{3}\right)=\left(\dfrac{2}{1}\right)\left(\dfrac{5}{3}\right)\left(\dfrac{1}{3}\right)=\dfrac{2\cdot5\cdot1}{1\cdot3\cdot3}=\dfrac{10}{9}.$

Example 2. $(a)\left(\dfrac{b}{c}\right)\left(\dfrac{d}{e}\right)\left(\dfrac{f}{g}\right)=\left(\dfrac{a}{1}\right)\left(\dfrac{b}{c}\right)\left(\dfrac{d}{e}\right)\left(\dfrac{f}{g}\right)=\dfrac{abdf}{ceg}.$

RULE 2. *To divide one fraction by another, multiply the first one by the reciprocal of the second.*

Example 1. $\frac{3}{5} \div \frac{4}{7} = \left(\frac{3}{5}\right)\left(\frac{7}{4}\right) = \frac{21}{20}.$

Example 2.

$$\frac{x}{x-1} \div \frac{x^2}{x^2-1} = \left(\frac{x}{x-1}\right)\left(\frac{x^2-1}{x^2}\right) = \frac{x(x-1)(x+1)}{x^2(x-1)}$$
$$= \frac{x+1}{x}.$$

Note. As in the cases of addition and subtraction, the rules for multiplication and division of fractions are postulates which may be observed to be true in all arithmetic cases. For example, since $\frac{8a}{2a} = 4$, where a can be anything except zero, $\frac{8 \text{ thirds}}{2 \text{ thirds}} = 4$. By rule 2, $\frac{8}{3} \div \frac{2}{3} = \left(\frac{8}{3}\right)\left(\frac{3}{2}\right) = 4$.

Also, $\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot bd}{\frac{c}{d} \cdot bd} = \frac{ad}{bc} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right).$

EXERCISE 12

Perform the indicated multiplications.

1. $(3)\left(\frac{15}{9}\right).$
2. $(5)\left(\frac{14}{25}\right).$
3. $(-2)\left(\frac{-4}{3}\right).$
4. $\left(\frac{-12}{35}\right)\left(\frac{14}{15}\right).$
5. $\left(\frac{-2}{3}\right)\left(\frac{9}{16}\right)\left(\frac{4}{15}\right).$
6. $\left(\frac{3}{4}\right)\left(\frac{8}{21}\right)\left(\frac{-12}{16}\right).$
7. $\left(\frac{-4}{5}\right)\left(\frac{-25}{24}\right)\left(\frac{9}{35}\right).$
8. $(5)\left(\frac{3}{7}\right)\left(\frac{-21}{15}\right).$
9. $\left(\frac{-7}{8}\right)\left(\frac{48}{84}\right)(-3).$
10. $(-11)\left(\frac{57}{110}\right)\left(\frac{-15}{19}\right).$
11. $(2x)\left(\frac{x-1}{x+2}\right).$
12. $(-3x)\left(\frac{2x+3}{9x-6x}\right).$

13. $\left(\frac{5x+2}{3x-1}\right)(3x-2)$.

14. $\left(\frac{3x-7}{2x+1}\right)(4x+2)$.

15. $\left(\frac{6x-5}{3x+2}\right)(6x+4)$.

16. $\left(\frac{15x-3}{5x-1}\right)(2x+3)$.

17. $\left(\frac{13x+3}{7x-1}\right)(21x-3)$.

18. $\left(\frac{17x-3}{15x-3}\right)(5x-1)(x)$.

19. $\left(\frac{13x-2}{21x-7}\right)(1-3x)(x)$.

20. $\left(\frac{15x-5}{1-3x}\right)(3x+1)(-3x)$.

21. $\left(\frac{7x-3}{2x^2-3x+1}\right)(1-x)(-3x)$.

22. $\left(\frac{6x+4}{6x^2+x-2}\right)(1-2x)(-5x)$.

23. $\left(\frac{6-10x}{5x^2+7x-6}\right)(x+2)(-x)$.

24. $\left(\frac{12-8x}{6x^2-19x+15}\right)(5-3x)\left(\frac{1}{4}\right)$.

25. $\left(\frac{16-4x}{3x^2-x-14}\right)(7-3x)\left(\frac{3}{2}\right)$.

26. $\left(\frac{6x^2-19x+10}{2-x-3x^2}\right)\left(\frac{x+1}{5-2x}\right)(3)$.

27. $\left(\frac{3x^2+x-2}{6x^2-13x+6}\right)\left(\frac{2-3x}{5-x}\right)(15)$.

28. $\left(\frac{3x-1}{2x-5}\right)\left(2-\frac{5}{x}\right)$.

29. $\left(\frac{3x-2}{2x+3}\right)\left(2+\frac{3}{x}\right)$.

30. $\left(\frac{2x-5}{3x-4}\right)\left(3-\frac{4}{x}\right)$.

31. $\left(\frac{5x-7}{6x-2}\right)\left(\frac{1}{x}-3\right)$.

32. $\left(\frac{4x+3}{5x-20}\right)\left(\frac{4}{x}-1\right)$.

33. $\left(\frac{3x-2}{x+3}\right)\left(\frac{2x-3}{x-3}\right)$.

34. $\left(\frac{5x+7}{x-5}\right)\left(\frac{2x-3}{x+7}\right)$.

35. $\left[\frac{x-(2x-3)}{x}\right]\left(\frac{-x}{2+3x}\right)$.

36. $\left[\frac{2x-(x+3)}{2x}\right]\left(\frac{x^2-3x}{x^2-9}\right)$.

EXERCISE 12a

Perform the indicated divisions.

1. $\frac{2x-3}{3x-4} \div (2x-3)$.

2. $\frac{3x-3}{5x+3} \div (x-1)$.

3. $\frac{4x+12}{7x+10} \div (x+3)$.

4. $\frac{2x-4}{2x+3} \div (x-2)$.

5. $\frac{6x+15}{3x+5} \div (2x+5)$.

6. $\frac{x^2-4}{x^2+4} \div (2-x)$.

7. $\frac{x^2-x-2}{x^2+x-2} \div (x+1)$.

8. $\frac{2x^2+3x-2}{2x^2-x-3} \div (x+2)$.

9. $\dfrac{12x^2 - 7x - 12}{12x^2 + 7x - 12} \div (3x - 1)$.

10. $\dfrac{3x^2 + 2x - 1}{3x^2 - 2x - 1} \div (3x - 1)$.

11. $(3x - 2) \div \dfrac{x - 2}{x + 3}$.

12. $(2x + 5) \div \dfrac{3x - 2}{x - 2}$.

13. $(5x - 3) \div \dfrac{x + 4}{2x - 5}$.

14. $(7x + 2) \div \dfrac{2 + 7x}{5 - 3x}$.

15. $(3x - 5) \div \dfrac{5 - 3x}{7x + 2}$.

16. $(6x + 5) \div \dfrac{6x^2 - x - 5}{6x^2 - 5x - 6}$.

17. $(3 + 2x) \div \dfrac{6x^2 + 5x - 6}{6x^2 - 11x - 2}$.

18. $\dfrac{(3x^2+2x-1)}{6x^2-7x-3} \div \dfrac{3x^2-4x+1}{2x^2-x-3}$.

19. $(2x^2 + 3x - 2) \div \dfrac{2x^2 - x - 3}{3x^2 + 2x - 1}$.

20. $(12x^2 - 7x - 12) \div \dfrac{12x^2 + 25x + 12}{6x^2 - x - 12}$.

21. $\dfrac{y - 3}{(x - 2)^2} \div \dfrac{3 - y}{x - 2}$.

22. $\dfrac{2x - 5}{3y + 7} \div \dfrac{5 - 2x}{14 + 6y}$.

23. $x \div \dfrac{1}{x}$.

24. $(y + 1) \div \dfrac{2}{3y}$.

25. $(2x - 1) \div \dfrac{1}{2}$.

26. $(3x + 2) \div \dfrac{2}{3}$.

27. $(6x^2 - 9x + 3) \div \dfrac{3}{4}$.

28. $(5x^2 + 10x - 5) \div \dfrac{5}{7}$.

29. $\dfrac{-(3x - 2)}{5x + 7} \div \dfrac{2 - 3x}{14 + 10x}$.

30. $\dfrac{x^2-3xy-4y^2}{2x^2+3xy-2y^2} \div \dfrac{-2y-x}{2x-y}$.

31. $\dfrac{1 - 9x^2}{x - 1} \div \dfrac{3x^2 + 2x - 1}{1 + 2x - 3x^2}$.

32. $\dfrac{y^2 - 2xy - 3x^2}{y^2 + 2xy - 3x^2} \div \dfrac{3x^2 + 5xy + 2y^2}{3x^2 + 4xy + y^2}$.

33. $\dfrac{2x^2 + xy - y^2}{2y^2 - 6xy + 4x^2} \div \dfrac{2x^2 + xy - y^2}{2x^2 - xy - y^2}$.

34. $\dfrac{6x^2 - 13xy + 6y^2}{xy - 3x + 2y - 6} \div \dfrac{3y^2 - 5xy + 2x^2}{6 - 2y + xy - 3x}$.

35. $\dfrac{2x^2 - 5xy + 2y^2}{2y^2 + 3xy - 2x^2} \div \dfrac{2y^2 - 3xy - 2x^2}{2x^2 - xy - y^2}$.

36. $\dfrac{xy - x - 2y + 2}{xy - 2x + y - 2} \div \dfrac{xy + x - 2y - 2}{xy + x + y + 1}$.

3.6. *Simplification of complex fractions*

We have seen that a fraction is complex if either the numerator or the denominator contains a fraction. In this case we shall speak of the *minor denominators*. For example, in the complex fraction $\dfrac{1+\dfrac{a}{2}+\dfrac{3}{b}}{\dfrac{1}{b}-\dfrac{1}{2}}$, the minor denominators are 2 and b.

To simplify a complex fraction we reduce it to a simple fraction in its lowest terms.

When neither the numerator A nor the denominator B of a complex fraction itself contains a complex fraction, there are two methods of simplifying $\dfrac{A}{B}$.

Method 1. Multiply both the numerator and the denominator by the L.C.M. of the minor denominators. (*See* Law 1, Art. 3.2.)

Example 1.

$$\frac{1+\dfrac{1}{2}+\dfrac{2}{3}-\dfrac{1}{12}}{\dfrac{5}{6}-\dfrac{2}{3}+1}=\frac{\left(1+\dfrac{1}{2}+\dfrac{2}{3}-\dfrac{1}{12}\right)12}{\left(\dfrac{5}{6}-\dfrac{2}{3}+1\right)12}$$

$$=\frac{12+6+8-1}{10-8+12}=\frac{25}{14}.$$

Example 2. $$\frac{\dfrac{a}{2}-\dfrac{b}{x}}{\dfrac{c}{x^2}+\dfrac{1}{2}}=\frac{\left(\dfrac{a}{2}-\dfrac{b}{x}\right)2x^2}{\left(\dfrac{c}{x^2}+\dfrac{1}{2}\right)2x^2}=\frac{ax^2-2bx}{2c+x^2}.$$

Method 2. Reduce the numerator and likewise the denominator to a simple fraction, and then apply Rule 2, *Art.* 3.5.

Example 1.

$$\frac{2+\dfrac{1}{2}-\dfrac{1}{3}}{3-\dfrac{1}{3}}=\frac{\dfrac{12}{6}+\dfrac{3}{6}-\dfrac{2}{6}}{\dfrac{9}{3}-\dfrac{1}{3}}=\frac{\dfrac{13}{6}}{\dfrac{8}{3}}=\left(\frac{13}{6}\right)\left(\frac{3}{8}\right)=\frac{13}{16}.$$

Example 2.

$$\frac{\dfrac{1}{x-1}+2}{1+\dfrac{1}{1-x^2}}=\frac{\dfrac{1}{x-1}+\dfrac{2(x-1)}{x-1}}{\dfrac{1-x^2}{1-x^2}+\dfrac{1}{1-x^2}}=\frac{\dfrac{2x-1}{x-1}}{\dfrac{2-x^2}{1-x^2}}$$

$$=\left(\frac{2x-1}{x-1}\right)\left(\frac{1-x^2}{2-x^2}\right)=\left(\frac{2x-1}{x-1}\right)\left(\frac{x^2-1}{x^2-2}\right)$$

$$=\frac{(2x-1)(x+1)}{x^2-2}.$$

In general, Method 1 is shorter and more efficient when the minor denominators are single terms; otherwise Method 2 is preferable.

When either the numerator or the denominator of a complex fraction is itself a complex fraction, the latter fraction must be simplified first. The same rule applies to the latter fraction and so on.

Example. $$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}}=\left\{\frac{1}{1+\left[\dfrac{1}{1+\left(\dfrac{1}{1+\dfrac{1}{x}}\right)}\right]}\right\}.$$

Here we simplify successively the fractions within parentheses, brackets, and braces, thus:

$$\left\{\frac{1}{1+\left[\dfrac{1}{1+\dfrac{x}{x+1}}\right]}\right\}=\left\{\frac{1}{1+\dfrac{x+1}{2x+1}}\right\}=\frac{2x+1}{3x+2}.$$

EXERCISE 13

Simplify the following fractions.

1. $\dfrac{\frac{1}{x}}{\frac{1}{y}}.$

2. $\dfrac{x+\frac{1}{y}}{x-\frac{1}{y}}.$

3. $\dfrac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

4. $\dfrac{\dfrac{1}{x+y}+1}{\dfrac{1}{x+y}-1}$.

5. $\dfrac{\dfrac{1}{x-y}-1}{\dfrac{1}{x+y}+1}$.

6. $\dfrac{\dfrac{x}{x+y}-1}{\dfrac{y}{x+y}}$.

7. $\dfrac{\dfrac{y}{y-x}-1}{\dfrac{x}{y-x}}$.

8. $\dfrac{\dfrac{a}{x}-\dfrac{b}{y}}{\dfrac{bx-ay}{xy}}$.

9. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}$.

10. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{x}{y}+\dfrac{y}{x}}$.

11. $\dfrac{\dfrac{6x^2-13xy+6y^2}{xy-3x+2y-6}}{\dfrac{3y^2-5xy+2x^2}{6-2y-3x+xy}}$.

12. $\dfrac{\dfrac{2x^2-5xy+2y^2}{2x^2-3xy-2y^2}}{\dfrac{2y^2-3xy-2x^2}{2x^2-xy-y^2}}$.

13. $\dfrac{\dfrac{2x^2+xy-y^2}{2y^2-6xy+4x^2}}{\dfrac{2y^2+xy-x^2}{2x^2-xy-y^2}}$.

14. $\dfrac{\dfrac{3x^2+2xy-y^2}{3x^2-2xy-y^2}}{\dfrac{4xy-3x^2-y^2}{4xy+3x^2+y^2}}$.

15. $\dfrac{\dfrac{2y^2-xy-3x^2}{2y^2-3xy+x^2}}{\dfrac{6x^2+5xy-6y^2}{6x^2-11xy-2y^2}}$.

16. $\dfrac{\dfrac{x^3-y^3}{(x-y)^3}}{\dfrac{x^2-y^2}{(x-y)^2}}$.

17. $\dfrac{\dfrac{-3(x-y)}{6(x-y)}}{\dfrac{5(x-y)^2}{3(x-y)^3}}$.

18. $\dfrac{1-\dfrac{x+y}{x}}{1-\dfrac{x-y}{x}}$.

19. $\dfrac{2-3\left(\dfrac{x-y}{y}\right)}{3+2\left(\dfrac{x+y}{y}\right)}$.

20. $\dfrac{3x-\dfrac{2-y}{x}}{\dfrac{3y-x}{x}-x}$.

21. $\dfrac{\dfrac{2x-y}{y-x}-(x-y)}{\dfrac{2y-x}{x-y}-(y-x)}$.

22. $\dfrac{\dfrac{3}{x}-4+x}{\dfrac{12}{x}-7+x}$.

23. $\dfrac{\dfrac{6}{x}+x-5}{1-\dfrac{x+2}{x^2}}$.

24. $\dfrac{\dfrac{-4}{x}-1+6x}{2-\dfrac{x+1}{x^2}}$.

25. $\dfrac{2-\dfrac{3x+2}{x}}{2-\dfrac{3x-1}{x}}$.

26. $2+\dfrac{3}{1+\dfrac{1}{x}}$.

27. $1-\dfrac{1}{1-\dfrac{1}{x}}$.

28. $\dfrac{1}{2x-1}-\dfrac{3}{1+\dfrac{1}{x}}$.

29. $\dfrac{2}{2x+1}+\dfrac{1}{2-\dfrac{1}{x+1}}$.

30. $\dfrac{1}{3x+x}-\dfrac{1}{3-\dfrac{1}{x+1}}$.

REVIEW EXERCISES

Reduce the fractions in 1–6 *to lowest terms.*

1. $\dfrac{4a^2+6ab-4b^2}{a^2-ab-6b^2}$.

2. $\dfrac{9x^2-12xy+4y^2}{27x^3-8y^3}$.

3. $\dfrac{(x+2y)^3+z^3}{(x+2y)^2-z^2}$.

4. $\dfrac{3mn+6kn+mc+2kc}{3mn+6kn-mc-2kc}$.

5. $\dfrac{p^4-q^4}{p^6-q^6}$.

6. $\dfrac{(2x-y)^3-5(2x-y)}{4x^2-y^2}$.

Carry out the indicated operations and express the answer to each as a simple fraction in lowest terms.

7. $\dfrac{3x-1}{3}-\dfrac{5x+2}{4}+\dfrac{x-1}{8}$.

8. $\dfrac{3y+a}{4y}-\dfrac{2y-a}{3y}-\dfrac{4}{3}$.

9. $\dfrac{3x-y}{4y^2}+\dfrac{2xy-1}{3xy}-\dfrac{2x+1}{4x^2}-\dfrac{1}{2x}$.

10. $-\dfrac{a^2+5b^2}{a^3-b^3}+\dfrac{a-b}{a^2+ab+b^2}-\dfrac{3}{a-b}.$

11. $\dfrac{3x}{x^2-y^2}+\dfrac{2x+1}{(x-y)^2}.$

12. $\dfrac{a}{(a-b)(b-c)}+\dfrac{b}{(c-a)(b-a)}+\dfrac{c}{(b-c)(a-c)}.$

13. $\dfrac{3x-5}{x-3}-2x-1.$

14. $\dfrac{x+1}{4x-4}-\dfrac{x+2}{3x+3}+\dfrac{x}{2(x+1)}.$

15. $\dfrac{4x-3}{x-2}-2.$

16. $1-\dfrac{x(x-y)}{1-xy}.$

17. $\dfrac{ab-3b^2}{3a-b}+a-2b.$

18. $\dfrac{x^2-y^2}{9z^2}\cdot\dfrac{18z}{(x-y)^3}.$

19. $\dfrac{a^2b^2+3ab}{9b^2-1}\cdot\dfrac{3b+1}{ab+3}.$

20. $\dfrac{x^2+xy-12y^2}{x^2-xy-6y^2}\cdot\dfrac{x^2-5xy-14y^2}{x^2+11xy+28y^2}.$

21. $\dfrac{x^3-x^2+4x-4}{2x^2-x}\cdot\dfrac{2x^2+x-1}{x^2+4}.$

22. $\dfrac{a(a+2b)}{a^4+4b^4}\cdot\dfrac{ab-2b^2}{a^2-4b^2}\div\dfrac{1}{a^2-2ab+2b^2}.$

23. $\left(\dfrac{a^3}{4}-2\right)\left(1+\dfrac{a}{2}\right).$

24. $\left(\dfrac{4x^2}{y^2}-1\right)\left(\dfrac{y}{2x-y}+1\right).$

25. $\left(\dfrac{3ab-b^2}{1-3ab}+1\right)\left(3a-\dfrac{a^2+ab}{1-b}\right).$

26. $\left(x-1+\dfrac{2}{x+1}\right)\left(x+1-\dfrac{3}{x-1}\right).$

27. $\dfrac{\dfrac{3x}{1-x}+1}{\dfrac{2}{1-x}}.$

28. $\dfrac{2-\dfrac{3}{a-1}}{\dfrac{3}{1-a}-1+\dfrac{2}{a^2-1}}.$

29. $\dfrac{x + \dfrac{3}{y - x}}{-\dfrac{1}{x - y} + \dfrac{2}{(y - x)^2}}$.

30. $\dfrac{\dfrac{m - n}{m - p} - \dfrac{m - p}{m - n}}{\dfrac{1}{m - p} - \dfrac{1}{m - n}}$.

31. $\dfrac{\dfrac{x^2}{y^2} - 1}{\dfrac{x^2}{y^2} - \dfrac{2x}{y} + 1}$.

32. $\dfrac{\dfrac{1}{a - b}}{1 + \dfrac{ab + 2b^2}{a^2 - b^2}}$.

33. $3 - \dfrac{1}{2 - \dfrac{1}{a + b}}$.

34. $3 - \dfrac{1}{2 - \dfrac{1}{a} - \dfrac{1}{b}}$.

35. $\dfrac{2(x^2 - 4)(2x - 3) - 2x(2x - 3)^2}{(2x - 3)^2}$.

36. $\dfrac{3(a^2 + 4)(a^2 + 5) - (a^2 + 5)^2}{(a^2 + 4)^2}$.

37. $\left(-\dfrac{2}{x + 1} + \dfrac{3}{x + 2} - \dfrac{1}{x + 3}\right) \cdot \dfrac{(x + 3)^2(x + 1)}{(x + 2)}$.

38. $\dfrac{(u + 2)^2(-3u^2) - 2u(u + 2)}{(u + 2)^3}$.

39. $\dfrac{a - b}{(a - b)^2 - c^2} - \dfrac{a + b}{(a - c)^2 - b^2}$.

40. $\dfrac{(a - b)^2 - c^2}{a^2 - (b - c)^2} \cdot \dfrac{(a + b)^2 - c^2}{a^2 - (b + c)^2}$.

41. $\left[\dfrac{a^3 - b^3}{a^3} \cdot \dfrac{b^2 - a^2}{(b - a)^2}\right] \div \dfrac{a^2b + ab^2 + b^3}{a^2 + 2ab + b^2}$.

42. $\dfrac{a + b}{a - b} - \dfrac{a - b}{a + b} - \dfrac{a - 2b}{b^2 - a^2}$.

43. $\left(1 - \dfrac{m^2}{n^2}\right)\left(\dfrac{n}{m - n} - 1\right) - \left(1 - \dfrac{m^3}{n^3}\right)\left(1 - \dfrac{mn + n^2}{m^2 + mn + n^2}\right)$.

44. $\left[\dfrac{(a + b)^2 - c^2}{a^2 + ab - ac} \cdot \dfrac{a^2b^2c^2}{a^2 + ab + ac}\right] \div \dfrac{a^2c^2}{abc}$.

Chapter Four

LINEAR EQUATIONS IN ONE UNKNOWN

4.1. *Identities and conditional equations*

An *equation* is an assertion that two expressions are equal. It is a sentence, in the language of algebra, in which the equality sign serves as the verb. The two expressions involved are called the *sides* or *members* of the equation.

An *identity* is an equation which is true for all values of the letters involved.*

Example 1. $(a + b)^2 = a^2 + 2ab + b^2$.

Example 2. $x^2 - y^2 = (x - y)(x + y)$.

When it is desired to emphasize the fact that an equation is an identity, the equality sign may be replaced by the symbol $\equiv$, read "is identically equal to." Thus

$$x - 1 \equiv -(1 - x).$$

A *conditional equation* is one which is not an identity. It is true for special values only, if any at all, of the letters involved.

Example. $5x - 3 = 2x + 9$.

A conditional equation may be translated into English as a question. The one in the example above asks, "What is

* More precisely, an identity is true for all values of the letters involved which make each side of the equation a definite number. For example,

$$\frac{3}{(x-1)(x+2)} = \frac{1}{x-1} - \frac{1}{x+2}$$

is an identity; but the values $x = 1$ and $x = -2$ are not permissible, since division by zero is not allowed.

the number such that 3 less than its product by 5 is 9 more than its double?"

Henceforth we shall use the word "equation" to mean "conditional equation."

While, as indicated above, an equation is in one sense an implied question, it is also, in many cases, a camouflaged answer (like the query, "Who wrote 'Gray's Elegy'?"). For as will be shown presently, when we apply a routine method to certain equations, the numbers which they ask about pop very readily out of hiding.

The letters representing the numbers or expressions sought are usually chosen from the last part of the alphabet (x, y, z, u, w, etc.) and are called *unknowns.*

4.2. *Integral rational equations*

An *integral rational* equation is one in which two integral rational polynomials in the unknowns are equated. Thus the unknowns appear either without exponents (since the exponent 1 is usually omitted) or with the exponents 2, 3, etc.

Example 1. $3x - 4 = 5x$. Here the unknown is x, and the value sought for it is -2, as may be verified by trial.

Example 2. $ay - by = c$. Here the unknown is y, which stands for the expression $\frac{c}{a - b}$, since this value substituted for y makes the equation an identity.

As we shall see, many practical problems lead to integral rational equations.

4.3. *Linear equations*

An integral rational equation in which the unknowns appear to the first degree in each term containing them is called a *first degree,* or *linear* * equation.

* We shall see that a first degree equation in not more than two letters may be represented graphically by a straight line — hence the adjective "linear."

Example 1. $3x - 7 = 4$. (Linear in x.)

Example 2. $5a^2y - 7b^2y = c$. (Linear in y.)

Example 3. $5x + 2y = 4$. (Linear in x and y.)

Hereafter in this chapter we shall consider only linear equations in one unknown.

4.4. *Solving a linear equation in one unknown*

When we *solve* an equation we find its *roots*, or the particular quantities which, when substituted for the unknown, make the two members equal. The root is said to *satisfy* the equation.

Example. The root of $5x - 3 = 2x + 9$ is 4, since $5 \cdot 4 - 3 = 2 \cdot 4 + 9$.

We shall now solve the equation

$$\frac{4x}{3} - 3 = \frac{x}{2} + 2 \tag{1}$$

by use of several axioms. (An *axiom*, as those who have studied geometry may recall, is a statement which seems to accord with experience, though we do not prove it, and which is often used in proofs of other statements. Thus a set of axioms, helps to serve as a foundation upon which one may build a system of mathematics.)

AXIOM 1. *When equals are multiplied by equals, the products are equal.*

Application. $\left(\frac{4}{3}x - 3\right)6 = \left(\frac{x}{2} + 2\right)6,$

or

$$8x - 18 = 3x + 12. \tag{2}$$

AXIOM 2. *When equals are added to equals, the sums are equal.*

Application. $(8x - 18) + 18 = (3x + 12) + 18,$

or

(3) $$8x = 3x + 30.$$

AXIOM 3. *When equals are subtracted from equals, the remainders are equal.*

Application. $8x - 3x = (3x + 30) - 3x,$

or

(4) $$5x = 30.$$

AXIOM 4. *When equals are divided by equals, the quotients are equal.*

Application. $$\frac{5x}{5} = \frac{30}{5},$$

or

(5) $$x = 6.$$

Check in (1): $\frac{4 \cdot 6}{3} - 3 = \frac{6}{2} + 2$, or $8 - 3 = 3 + 2$.

It should be noted that the root, 6, satisfies each of equations (1) to (5). A group of equations, such as (1) to (5), which are satisfied by the same value of the unknown, are called *equivalent.*

An equation is solved, then, by finding a succession of equivalent equations, the last one of which exposes the root of all of them.

The simple rule that a term may be *transposed* from one member of the equation to the other by changing its sign may be used in practice in place of Axioms 2 and 3; but the reason for the rule should be understood.

The essential process in solving any linear equation may be described as follows:

Step 1. Simplify all complex fractions involved.

Step 2. Remove parentheses, carrying out indicated multiplications.

Step 3. Clear the equation of fractions by multiplying both members by the L.C.M. of the denominators.

Step 4. Transpose the terms containing the unknown to the left side of the equality sign, and all other terms to the right side.

Step 5. Factor the left member, expressing it as the product of the unknown by a second factor. The latter is called the *coefficient of the unknown.*

Step 6. Divide both members of the equation by the coefficient of the unknown.

For example, applying these steps to the equation

$$\frac{3\left(\frac{x}{2} + 3\right)}{\frac{1}{2} + 2} = \frac{6x}{7} + 4, \tag{6}$$

we get in succession the following equivalent equations:

$$\frac{3(x + 6)}{1 + 4} = \frac{6x}{7} + 4; \tag{7}$$

$$\frac{3x + 18}{5} = \frac{6x}{7} + 4; \tag{8}$$

$$21x + 126 = 30x + 140; \tag{9}$$

$$21x - 30x = 140 - 126; \tag{10}$$

$$x(-9) = 14 \tag{11}$$

$$x = \frac{14}{-9} = -1\tfrac{5}{9}. \tag{12}$$

The student can easily avoid blunders which are often made if he will keep in mind the following simple rule: *Whatever is done to the left member must be done to the right member.*

For example, if $2x = 5$, we should be wrong in concluding that $x = 5 - 2 = 3$, because the rule does not permit us to *divide* the left member by 2 and to *subtract* 2 from the right member, as is done in the foregoing incorrect "solution."

EXERCISE 14

Remove all signs of aggregation according to the rules, multiplying factors as indicated, and then determine which of the following equations are identities and which are conditional equations. Solve the latter.

1. $3x = 15.$ **2.** $5x + 20 = 0.$

3. $x(x - 1) = x^2 - x.$ **4.** $7x - 10 = x + 2.$

5. $2x^2 - 3x = x(2x - 3).$ **6.** $4x - 3 = 5x - 5.$

7. $(x - 1)^2 = x^2 - 2x + 1.$ **8.** $3 - 2x = 7 + 2x.$

9. $5x^2+3x-1=x(5x+3)-1.$ **10.** $1 - 3x = 5x + 4.$

11. $x + 2 + 3x - 1 = 4x + 1.$ **12.** $3 + 4x = 6x - 7.$

13. $\dfrac{\dfrac{x}{2} - \dfrac{x}{3}}{\dfrac{1}{2} - \dfrac{3}{4}} = \dfrac{1}{4}.$ **14.** $\dfrac{2x - \frac{3}{4}}{2} = 2 + \dfrac{3x}{4}.$

15. $x - \dfrac{2x - \frac{1}{3}}{4} = 5.$ **16.** $3x - 10 = 10 - 3x.$

17. $1.5x - .7x = .6x + .7.$ **18.** $.5x + .07 = .8 + .06x.$

19. $3x + (5 - 2x) = 4x - 5.$

20. $7 - (5x + 3) + 6x = 10 - (3x + 5).$

21. $8x - (3 + 4x) + 5 = 7x - (3 + x).$

22. $(9 - 7x) + (3 + x) = (15 - 8x) + 4.$

23. $10x + 5 - (3x - 4) = 4x - 3.$

24. $-2(5x - 7) + (3x + 2) = 4x - 2(3 + x).$

25. $5(2 + x) - [3(7 - 3x) - 3] = 7x + 5.$

26. $5(2 + x) - [3(x - 7) + 4(2x - 3)] = x.$

27. $8(2 - x) - 5(3 + x) = -[2(x - 1) + 3x].$

28. $7(x - 3) - 2(3 + x) = 5[(1 - 3x) - 4].$

29. $x - 2 - \dfrac{3x - 2}{2} = \dfrac{1}{2} + x.$ **30.** $\dfrac{x + 2}{2} - \dfrac{3x - 2}{2} = \dfrac{x}{2} + 1.$

31. $\dfrac{2x - 3}{3} - \dfrac{3x - 2}{3} = \dfrac{2x + 2}{\frac{3}{4} - 2}.$ **32.** $4 - 3x + \dfrac{2x - 3}{5} = \dfrac{4x + 5}{2}.$

33. $\dfrac{\dfrac{3x - 2}{4}}{2} - \dfrac{\dfrac{2x - 3}{4}}{3} = \dfrac{\dfrac{3x + 5}{4}}{6}.$ **34.** $\dfrac{2(2x-5)}{3} - \dfrac{\dfrac{6x+5}{2}}{3} = \dfrac{4x-7}{6}.$

35. What number added to 3 is twice as much as the number minus 1?

36. If when 1 is subtracted from 3 times a certain number, the result is 5 more than the original number, what is that number?

37. If a number is multiplied by 5, the product is 2 less than twice the sum of the number and 7. Find the number.

38. One child is twice as old as another. Two years ago he was 3 times as old. Find their ages.

39. Fred has 3 times as many marbles as Bob. If he gives Bob one he will have twice as many. How many has each?

40. The quotient obtained by dividing a certain number by .3 is 5 less than twice the number. Find the number.

41. A man invested one sum at 6% and double that sum at 5%, receiving $80.00 interest in one year. How much was invested at each rate?

42. An airplane traveling 200 miles an hour starts at noon after a transport which left at 10 A.M. and travels 150 miles per hour. When will the plane overtake the transport?

4.5. *Operations which yield equivalent equations*

When terms are transposed in an equation, or when both members are multiplied or divided by the same number, excluding zero, the resulting equation is equivalent to the first one. All of the operations required in Exercise 14 were of this nature.

4.6. *Operations which may not yield equivalent equations*

Multiplication of both members of an equation by zero yields the identity $0 = 0$. While this of course is true, it is not equivalent to the first equation. Division of both members by zero is not allowable, as we have seen.

Again, when we multiply both members of an equation by an expression containing the unknown, the new equation may have roots not satisfying the first one. These roots are called *extraneous*.

Example 1. The equation $x - 2 = 0$ has the root 2. Multiplying both members by $x - 3$, we have: $(x - 2)(x - 3) = 0(x - 3) = 0$. This has the roots 2 and 3; but 3 does not satisfy the first equation and hence is extraneous.

Example 2. Consider the equation

$$\frac{1}{(x-1)(x-2)} = \frac{2}{x-2} - \frac{1}{x-1}.$$

Multiplying both members by $(x - 1)(x - 2)$ to clear fractions, we get $1 = 2(x - 1) - (x - 2)$. This has the root 1. But 1 does not satisfy the first equation, since it calls for the non-permissible division by zero. Hence 1 is extraneous, and the first equation has no root.

Finally, if we divide both members by an expression containing the unknown, we usually *lose* one or more roots of the original equation.

Example. The equation $x(x - 1) = 3(x - 1)$ is satisfied by $x = 1$ or $x = 3$, as may be verified by substitution. Dividing both members by $x - 1$, we get $x = 3$, the root 1 being lost.

Summing up, we find that *when the members of an equation are multiplied or divided by zero or an expression containing the unknown, the resulting equation may not be equivalent to the first one.*

When operations of this sort are necessary in solving equations, as in Example 2 above, the solution is not complete until the roots found have been tested in the original equation, and the extraneous roots have been rejected.

EXERCISE 15

The following equations may be reduced to linear ones and then solved. Where extraneous roots may have been introduced the answers must be tested.

1. $\dfrac{1}{x-1} + \dfrac{2}{x+1} = \dfrac{3}{x-1}.$ 2. $\dfrac{3}{x-2} + \dfrac{1}{x+2} = \dfrac{8}{x^2-4}.$

3. $\dfrac{3}{x-3}-\dfrac{1}{x+2}=\dfrac{5}{x^2-x-6}.$

4. $\dfrac{4}{x+2}-\dfrac{3}{x+3}=\dfrac{8}{x^2+5x+6}.$

5. $\dfrac{3}{x+3}+\dfrac{4}{x-4}=\dfrac{28}{x^2-x-12}.$

6. $\dfrac{2x-1}{3x+2}=\dfrac{1}{2}.$

7. $\dfrac{5x+3}{2x-1}=\dfrac{1}{3}.$

8. $\dfrac{3-2x}{5-4x}=\dfrac{2}{3}.$

9. $\dfrac{3x+5}{6x-7}=\dfrac{3}{4}.$

10. $\dfrac{5-4x}{2-3x}=\dfrac{5}{4}.$

11. $\dfrac{3}{x}-\dfrac{x-2}{x-3}=\dfrac{4-x}{x-3}.$

12. $\dfrac{2}{x}-\dfrac{3x+2}{x+1}=\dfrac{2-3x}{x+1}.$

13. $\dfrac{2}{x}+\dfrac{x+3}{x-2}=\dfrac{x+5}{x-2}.$

14. $\dfrac{3x-5}{3x-2}=\dfrac{x+7}{x+3}.$

15. $\dfrac{3x-4}{3x+1}=\dfrac{2x+5}{2x-1}.$

16. $\dfrac{5x+3}{2x-1}=\dfrac{4+5x}{2x-3}.$

17. $\dfrac{4x+1}{2x+3}=\dfrac{2x-1}{x+2}.$

18. $\dfrac{6x+5}{2x+7}=\dfrac{3x-6}{x+3}.$

19. $\dfrac{3-x}{x-4}=\dfrac{2}{x}-\dfrac{5+x}{x-4}.$

20. $\dfrac{2x-5}{x+2}=\dfrac{7}{x}+\dfrac{2x+3}{x+2}.$

21. $\dfrac{x-1}{2}-\dfrac{x+2}{2}+\dfrac{3x-1}{2}=-\dfrac{1}{2}.$

22. $\dfrac{3x+2}{5}-\dfrac{x-3}{5}+\dfrac{2}{5}=\dfrac{4x+9}{5}.$

23. $\dfrac{2x-5}{3}+\dfrac{4}{3}-\dfrac{x+3}{3}=-\dfrac{x}{3}.$

24. $\dfrac{4x-3}{6}-\dfrac{5}{6}-\dfrac{6x-3}{6}=-2x.$

25. $\dfrac{3x+1}{7}+\dfrac{5}{7}-\dfrac{6x-2}{7}=3x+1.$

26. $\dfrac{2}{x-3}-\dfrac{3}{x+3}=\dfrac{5}{x^2-9}.$

27. $\dfrac{3x+1}{x+2}-\dfrac{2x-1}{x-3}=\dfrac{x^2+7}{x^2-x-6}.$

28. $\dfrac{3}{x-5}+\dfrac{2}{x+3}=\dfrac{5-x}{x^2-2x-15}.$

29. $\dfrac{5}{x+5} - \dfrac{4}{x+1} = \dfrac{3x+10}{x^2+6x+5}.$

30. $\dfrac{4}{2x-3} - \dfrac{5}{3x-2} = \dfrac{5x-7}{6x^2-13x+6}.$

31. $\dfrac{1}{3x-2} + \dfrac{2}{4x-1} = \dfrac{5x-7}{12x^2-11x+2}.$

32. $\dfrac{3}{5x+1} - \dfrac{2}{x-1} = \dfrac{5x-7}{5x^2-4x-1}.$

33. $\dfrac{2}{3x+1} - \dfrac{4}{2x-1} + \dfrac{10}{6x^2-x-1} = 0.$

4.7. *Literal coefficients*

If P is the perimeter of a rectangle of length L and width W, the relation between P, L, and W is evidently expressed by the formula:

(1) $$P = 2L + 2W.$$

Solving (1) for L, we have $-2L=2W-P$; $2L=P-2W$;

(2) $$L = \frac{P-2W}{2}.$$

Similarly,

(3) $$W = \frac{P-2L}{2}.$$

Results (1), (2), and (3) are different versions of the same equation, solved respectively for P, L, and W in terms of the remaining letters. In such equations any one of the letters may be considered as the unknown. Thus, if the perimeters and widths of many different rectangles were given, and we were asked to get the various lengths, the most efficient way would be to use (2) as a formula. If $P = 100$ and $W = 6$, $L=\dfrac{100-2\cdot 6}{2}=44$; if $P=90$ and $W=10$, $L=\dfrac{90-2\cdot 10}{2}=35$, etc. Similarly (1) and (3) are formulas for P and W respectively.

Whenever an equation contains other letters than the unknown, those letters will usually appear in the root of the equation. For example, given

$$Ax + B = 0, \tag{4}$$

the root is evidently $-\dfrac{B}{A}$.

Such equations are said to have *literal* coefficients. It should be understood that the coefficients in an equation include not only multipliers of the unknown, as A in (4), but also the terms, such as B in (4), which do not contain the unknown.

Any linear equation in the unknown x may be represented by (4). For instance, in

$$2ax - 3y = 0, \tag{5}$$

the A of (4) stands for $2a$, and the B for $-3y$. Since the root of (4) is $-\dfrac{B}{A}$, which represents a single number for any given set of values for the literal coefficients, we can conclude that:

A linear equation has exactly one root.

Clearly an equation with literal coefficients represents infinitely many particular equations with numerical coefficients. Special cases of (1), for example, are $3x + 2 = 0$, $5x - 17 = 0$, etc. An equation like (4) is said to be more *general* than one with numerical coefficients. In algebra, and in fact in all mathematics, it is often desirable to have problems and solutions as general as possible, thus covering many cases in a single operation.

In solving an equation with literal coefficients, the steps of Art. 4.4 may still be used.

Example. Solve

$$\frac{\frac{ax}{2}}{b} - \frac{c}{3d} + x + 1 = 2. \tag{6}$$

Solution.

Step 1. $\dfrac{ax}{2b} - \dfrac{c}{3d} + x + 1 = 2.$

Step 2. Not needed in this case.

Step 3. $\left(\dfrac{ax}{2b} - \dfrac{c}{3d} + x + 1\right) 6bd = 2(6bd),$

or $\qquad 3axd - 2bc + 6bdx + 6bd = 12bd.$

Step 4. $3axd + 6bdx = 2bc - 6bd + 12bd.$

Step 5. $x(3ad + 6bd) = 2bc + 6bd.$

Step 6. $x = \dfrac{2bc + 6bd}{3ad + 6bd},$ or $\dfrac{2b(c + 3d)}{3d(a + 2b)}.$

A complete check, of course, would require the substitution of the literal root in the original equation. By way of a brief and practical partial check, however, we may substitute specific numbers for a, b, c, and d. For instance, if $a = b = c = d = 1$, then (6) becomes $\dfrac{x}{2} - \dfrac{1}{3} + x + 1 = 2$, with the root $\dfrac{8}{9}$. Also $\dfrac{2bc + 6bd}{3ad + 6bd}$ becomes $\dfrac{8}{9}$. Or let $a = c = 0$. Then $x + 1 = 2$ or $x = 1$, while the root being tested becomes $\dfrac{6bd}{6bd} = 1.$

The partial check which is shortest, when permissible, is to replace all letters except the unknown by zero. Why not set $a = b = c = d = 0$ in the case above?

EXERCISE 16

Solve the following equations with literal coefficients. The unknown to be solved for is in each case x, y, z, u, v, or w.

1. $\dfrac{3x - a}{b} = 3 + \dfrac{cx}{2}.$

2. $\dfrac{\left(2a + \dfrac{c}{2}\right)x}{d} - \dfrac{3b}{5} - 2x = 0.$

3. $cy - d = a - cy.$

4. $cw - bd = bc - dw.$

5. $ac - dv = cv - ad.$

6. $au - bd = bu - ad.$

7. $ac - bx = bc - ax.$

8. $cy - ad = \dfrac{bc - dy}{m + n}.$

9. $\dfrac{u + n}{a - b} = \dfrac{u - n}{a + b}.$

10. $\dfrac{m + v}{a - b} = \dfrac{m - v}{a + b}.$

11. $\dfrac{m + n}{w - b} = \dfrac{m - n}{w + b}.$

12. $\dfrac{m + n}{a - x} = \dfrac{m - n}{a + x}.$

13. $\dfrac{2r - n}{y + 2b} = \dfrac{r + 2n}{y - 2b}.$

14. $\dfrac{2r - n}{a + 2z} = \dfrac{r + 2n}{a - 2z}.$

15. $\dfrac{2u - n}{a + 2b} = \dfrac{u + 2n}{a - 2b}.$

16. $\dfrac{2r - v}{a + 2b} = \dfrac{r + 2v}{a - 2b}.$

17. $\dfrac{w - 2r}{2m - 3n} = 2rw.$

18. $\dfrac{2m - 3x}{2m + 3x} = m + n.$

19. $y = \dfrac{2y - 3a}{b}.$

20. $\dfrac{z - 2r}{a - 2b} = az.$

Solve each of the following formulas for the letters indicated:

21. $l = a + (n - 1)d$, for a; for n; for d.

22. $E = I\left(R + \dfrac{r}{n}\right)$, for r; for R; for n.

23. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}$, for q; for p; for f.

24. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$, for R_1; for R_2; for R.

25. $P = A(1 - dt)$, for d; for A; for t.

26. $\dfrac{a}{v} = \dfrac{m}{M + m}$, for m; for M; for v.

27. $S = \dfrac{rl - a}{r - 1}$, for l; for a; for r.

28. $F = \dfrac{KmM}{d^2}$, for m; for M.

29. $v = v_0 + gt$, for t; for v_0; for g.

30. $s = \frac{1}{2}at^2$, for a.

31. $A = \frac{1}{2}h(a + b)$, for h; for b.

32. $V = r^2(a - b)$; for a.

33. $\dfrac{W - W_1}{W - W_2} = a$, for W; for W_2.

34. One boy is $2a$ years old and a second boy is $2b$ years old. What is their average age?

35. One child is a years old, and a second one is $2b$ years older than the first. What is their average age?

36. Henry had a marbles. After buying 10 more he had half as many as John. How many had John?

37. A boy who had x dimes in his pocket found half a dollar and then gave half of the money he had with him to his mother. What was the value of her share in cents?

38. A man who owed a dollars paid x dollars on the debt and then owed $\frac{1}{3}$ of the original amount. Find x.

39. In making a trip a man averaged a miles per hour going and b miles per hour on his return. If he was c hours on the road, how far from home did he go?

40. How many pounds of cream testing $x\%$ butter fat must be added to y pounds of milk testing $z\%$ butter fat to give milk testing $w\%$ butter fat?

41. If oranges cost c cents per dozen, how many oranges can one buy for a dollars?

EXERCISE 17

Solve the following stated problems involving fractions.

1. If a certain number is divided by 4 the Quotient is 3 and the remainder is 3. Find the number.

Note. In this and succeeding problems we use the relation (Art. 1.10)

$$\frac{\text{dividend}}{\text{divisor}} = \text{Quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Observe the distinction here made between "Quotient" and "quotient." For example, the Quotient of 13 divided by 4 is 3, while the quotient, or total result of the division, is $3\frac{1}{4}$.

2. What number is divided by 7 if the Quotient is 3 and the remainder is 2?

3. If 2 is added to a certain number and this sum is divided by 3 the Quotient is 4 and the remainder is 2. Find the number.

4. One number is 3 more than another. If the first one is divided by the second the quotient is $\frac{3}{2}$. Find the numbers.

5. One number is 2 less than another. If the first one is divided by the second the quotient is $\frac{2}{3}$. Find the numbers.

6. The difference between two numbers is 3. If the larger is divided by the smaller the quotient is $\frac{4}{3}$. Find the numbers.

7. If x is added to both the numerator and denominator of $\frac{1}{5}$, the new fraction formed equals $\frac{1}{2}$. Find the value of x.

8. Find the number which must be added to both the numerator and denominator of $\frac{1}{3}$ to make the new fraction equal to $\frac{3}{4}$.

9. What number must be subtracted from both the numerator and denominator of $\frac{15}{17}$ to yield the fraction $\frac{3}{5}$?

10. If a certain number is added to the numerator and subtracted from the denominator of $\frac{1}{8}$, the new fraction equals $\frac{4}{5}$. Find the number.

11. The numerator of a fraction is 2 less than the denominator. If 1 is added to both numerator and denominator, the resulting fraction is $\frac{2}{3}$. Find the original fraction.

12. The denominator of a fraction is 2 more than the numerator. If 11 is added to both numerator and denominator, the new fraction can be reduced to $\frac{14}{15}$. Find the original fraction.

13. The denominator of a fraction is 1 more than 3 times the numerator. If 3 is added to both numerator and denominator, the new fraction equals $\frac{1}{2}$. Find the original fraction.

4.8. *Geometric problems*

Many problems that are geometric in character occur in one's normal experience. They vary widely and include lengths, areas, volumes, relative sizes of angles, etc.

Example 1. A 5-foot string is cut into two pieces, one of which is $\frac{4}{5}$ of the other. Find the length of each piece.

Solution.

Let $\quad x =$ no. ft. in the longer piece.

Then $\dfrac{4x}{5} =$ no. ft. in the shorter piece.

(1) $$x + \frac{4x}{5} = 5,$$

since the sum of the parts of anything equals the whole of it. Solving (1), we have

$$x = \frac{25}{9}, \text{ the greater length;}$$

$$\frac{4x}{5} = \frac{20}{9}, \text{ the smaller length.}$$

Example 2. Two rectangles have the same width. One is 2 inches longer than the other and 5 inches longer than its width. If the difference in their areas is 10 square inches, find the dimensions of each.

Solution.

Let $x =$ the width of each rectangle in inches.

$x + 5 =$ the length of the longer rectangle, in inches.

$x + 3 =$ the length of the shorter rectangle, in inches.

$$x(x + 5) - x(x + 3) = 10.$$

Solving, we have $x = 5$; $x + 5 = 10$; $x + 3 = 8$.

Hence the rectangles are 10 by 5 and 8 by 5 inches respectively.

Note. The area of a circle of radius r is πr^2 square units. The volume of a right circular cylinder of radius r and height h is $\pi r^2 h$ cubic units.

Example 3. Each of two right circular cylinders has an altitude of 10 inches. The radius of the base of the larger is two inches greater than that of the other. The difference in their volumes is 80π cubic inches. Find the radius of the base of each cylinder.

Solution.

Let x = radius of base of smaller cylinder, in inches.
$x + 2$ = radius of base of larger cylinder, in inches.
$10\pi x^2$ = volume of the smaller cylinder, in cubic inches.
$10\pi(x + 2)^2$ = volume of the larger cylinder, in cubic inches.

$$(2) \qquad 10\pi(x + 2)^2 - 10\pi x^2 = 80\pi.$$

Solving (2), we find that $x = 1$ and $x + 2 = 3$.

Example 4. The first angle of a triangle is equal to $\frac{1}{3}$ of the second angle and is also equal to $\frac{1}{2}$ of the third angle. Find the three angles.

Solution.

Let x = no. degrees in the first angle.
Then $3x$ = no. degrees in the second angle.
$2x$ = no. degrees in the third angle.

$$x + 3x + 2x = 180.$$

Solving, we have $x = 30$, $3x = 90$, $2x = 60$.

EXERCISE 18

Form algebraic equations and solve the following problems.

1. A 7-foot cord is cut into two pieces so that one piece is $\frac{2}{3}$ as long as the other. Find the length of each piece.

2. A piece of wire 4 feet long is cut and the pieces are bent to form two circles. If the diameter of one circle is $\frac{3}{5}$ that of the other, find the length of each piece.

3. One string is 3 inches longer than another. If their combined length is 17 inches, find the length of each.

4. A 26-inch cord is cut into three pieces. The first is $\frac{3}{4}$ as long as the second, and the second is $\frac{2}{3}$ as long as the third. How long is each?

5. Some 12-foot boards are to be cut into two pieces to make lids for two boxes, one of which is $\frac{5}{7}$ as long as the other. How should each board be divided?

6. The width of a rectangle and the side of a square are equal. If the length of the rectangle is 5 inches more than its width, and

its area is 50 square inches more than that of the square, find the dimensions of each.

7. The radius of one circle is 3 inches more than that of another, and their areas differ by 27π square inches. Find the radius of each.

8. The diameter of one circle is 4 inches more than that of another, and its area is greater by 12π square inches. Find the diameter of each.

9. Two right triangles have equal bases. The two sides of each triangle form its base and altitude. One triangle is isosceles but the altitude of the other is 4 inches longer than its base. If their areas differ by 12 square inches, find the dimensions of each.

10. The altitudes of two triangles and the base of one of them are equal. The base of the other is 6 inches more than its altitude, and the difference in the areas is 21 square inches. Find the base and altitude of each.

11. The radius of the base of one right circular cylinder is 3 inches more than that of another, and the altitude of each is 15 inches. If the difference in their volumes is 225π cubic inches, find the radius of the base of each.

12. One of the acute angles of a right triangle is 10° less than the other. Find the number of degrees in each angle.

13. The first angle of a triangle equals $\frac{1}{2}$ of the second, and the third equals the sum of the first and second. Find each angle.

14. The vertical angle of an isosceles triangle is 20° more than the sum of the equal angles. Find each angle.

15. Work problem 14 with "20° more" replaced by "4° less."

16. The first angle of a triangle is $\frac{1}{3}$ as large as the second and $\frac{1}{5}$ as large as the third. Find each angle.

4.9. *Problems involving time, rate, and distance*

Many problems are solved by means of the formula

$$d = rt, \tag{1}$$

where d represents the number of units of distance, r the number of units of distance traveled in one unit of time, and t the number of units of time. For brevity, d, r, and t are

called *distance*, *rate*, and *time*. It must be understood, however, that (1) holds good *only* if the rate is constant, such as it is, for instance, in a car whose speedometer needle remains fixed at the 40 m.p.h. (miles per hour) position.

Equation (1) is used, of course, to find the distance when the rate and time are known. Solved for r and t, it takes the two alternate forms,

$$(2) \qquad r = \frac{d}{t},$$

and

$$(3) \qquad t = \frac{d}{r},$$

used respectively to find r and t when the other two quantities are known.

EXERCISE 19

Form equations and solve the following problems.

1. If one car runs 40 m.p.h. and another car runs 50 m.p.h., in how many hours will the sum of their distances be 300 miles?

Let $x =$ no. hrs. required.
Then $40x =$ no. mi. first car runs. (Here $rt = 40x$.)

$50x =$ no. mi. second car runs.

$$40x + 50x = 300.$$
$$90x = 300.$$
$$x = \tfrac{300}{90} = \tfrac{10}{3}.$$

2. A car running north at 45 m.p.h. passes point A at 12 o'clock. A second car running north at 50 m.p.h. passes A at 1:30 o'clock. When will the second car overtake the first one?

3. A is 450 miles west of B. A car starts east from A at 40 m.p.h., and at the same time another starts west from B at 45 m.p.h. In how many hours will they meet?

4. Two men are 400 yards apart and walk straight toward each other. If one walks 80 y.p.m. (yards per minute) and the other 90 y.p.m., in how many minutes will they meet?

5. Two men run in the same direction around a 440-yard track. If one runs 10 y.p.s. (yards per second) and the other $\frac{4}{5}$ as fast, in how many seconds will the faster one gain a lap?

6. One man can run 9 y.p.s. and the other can run 8 y.p.s. If they start at the same time and run in opposite directions around a 440-yard track, in how many seconds will they meet?

7. A car starts east at 40 m.p.h. A second car starts east from the same point one hour later at 50 m.p.h. When will the second car be 10 miles ahead of the first one?

8. A car starts south at 40 m.p.h. One hour later a second car starts south from the same point. If the second car overtakes the first one in three hours, find its speed.

9. A man walks toward a certain town at 2 m.p.h. One hour later a second man starts from the same place at 3 m.p.h. He overtakes the first man just as he reaches the town. How far was it to town and how long did each man require for the trip?

10. A train that runs 50 m.p.h. passes a station 2 hours behind a slower train and overtakes it in 3 hours. Find the speed of the slower train.

11. The current in a certain stream flows 3 m.p.h. A crew can row twice as fast downstream as it can row upstream. How fast can it row in still water?

Let $x =$ no. m.p.h. rowed in still water.

Then $x + 3 =$ no. m.p.h. rowed downstream.

$x - 3 =$ no. m.p.h. rowed upstream.

$$x + 3 = 2(x - 3).$$

Solving, we have

$$x = 9.$$

12. The current in a stream flows 2 m.p.h. A crew can row $\frac{3}{2}$ times as fast downstream as it can row upstream. How fast can it row in still water?

13. The current in a stream flows 4 m.p.h. A crew can row $\frac{2}{3}$ as fast upstream as it can row downstream. How fast can it row in still water?

14. The rate of the current is 3 m.p.h. A crew can row 9 miles downstream in the time required for rowing 4 miles upstream. How fast can it row in still water?

15. Work problem 14 if the current flows 2 m.p.h. and the downstream and upstream distances are respectively 18 and 6 miles.

16. A man can row 4 m.p.h. in still water. He can row 12 miles downstream in the time required for rowing 4 miles upstream. Find the rate of the current.

17. An airplane has a cruising speed of 300 m.p.h. On a certain day its ground speed when traveling with the wind was twice its speed against the wind. How fast was the wind blowing?

18. On a certain day the wind at 5000 feet was blowing 50 m.p.h., and at 10,000 feet it was blowing 20 m.p.h. in the same direction. A plane flew at 5000 feet with the wind from A to B in 3 hours, and returned at the 10,000 feet level in 5 hours. Find its speed in still air and the distance from A to B.

4.10. *Problems concerning money*

Most practical problems about money deal with principal, interest, rate of interest, wages, profit and loss. In some other problems, usually less practical in nature, the object is to find the number or denomination of pieces of money involved.

The *principal* is the sum of money that bears interest.

The *rate* is the fraction of the principal paid for its use during a certain period of time — usually a year. Stated in percentage, it is this fraction multiplied by 100. Thus, the rate of $\frac{06}{100}$ or .06 becomes "6%."

The *time* is the interval during which the principal is used.

The *interest* is the total sum paid for use of the principal.

The *amount* is the sum of the principal and interest.

In formulas the following notation is customary.

A represents the number of dollars in the amount.
P represents the number of dollars in the principal.
I represents the number of dollars in the interest.
t represents the number of interest periods in the time interval.
r represents the rate, written as a decimal fraction.

We shall use the following formulas.

(1) $$I = Prt.$$

Solving (1) for P, r, and t successively, we have

(2) $$P = \frac{I}{rt},$$

(3) $$r = \frac{I}{Pt},$$

and

(4) $$t = \frac{I}{Pr}.$$

(5) $A = P + I$, from the definition.

From (5) and (1),

(6) $$A = P + Prt = P(1 + rt).$$

When there are two or more investments, subscripts may be used to distinguish between them. For example, P_1, read "P sub-one," means the first principal, P_2 the second principal, etc. Similarly, I_1 means the interest on P_1; I_2 is the interest on P_2, etc.

Example 1. A man invests \$6000, part at 5% and part at 6%. The total interest for one year is \$320. How much was each investment?

Solution. Let the two investments be P_1 and $6000 - P_1$. For the first investment, $r = .05$ and $t = 1$; for the second, $r = .06$ and $t = 1$.

Substituting in (1) we have

$$I_1 = P_1(.05)(1) = .05P_1,$$

and $$I_2 = (6000 - P_1)(.06)(1) = 360 - .06P_1.$$

But $$I_1 + I_2 = 320, \text{ the total interest.}$$

Hence

$$.05P_1 + (360 - .06P_1) = 320.$$

Solving, we have

$$P_1 = 4000,$$

and $$6000 - P_1 = 2000.$$

Example 2. What principal will yield \$24 interest in 2 years at 6%?

Solution. Here $I = 24$, $r = .06$, $t = 2$, and P is the unknown. Using these values in (2), we have

$$P = \frac{24}{(.06)(2)} = \frac{24}{.12} = \frac{2400}{12} = 200.$$

Example 3. Two men work 6 days and receive \$54 as wages. If one receives $\frac{4}{5}$ as much per day as the other, find the daily wage of each.

Solution. Let x = the daily wage of the second man in dollars.

Then $\frac{4x}{5}$ = the daily wage of the first man in dollars.

$$x + \frac{4x}{5} = \frac{54}{6} = 9.$$

Solving, we have

$$x = 5,$$
$$\frac{4x}{5} = 4.$$

EXERCISE 20

1. How much interest will be earned by an investment of \$300 at 5% for 2 years?

2. An investment of \$200 yields \$8 interest in a year. Find the rate.

3. At what rate will \$250 yield \$20 interest in 2 years?

4. At what rate will \$3500 yield \$350 in 3 years?

5. At what rate will \$4700 yield \$282 in 2 years?

6. Find the time required for \$6500 to yield \$650 at 5%.

7. A sum of \$7500 is invested, part at 6% and part at 5%. The annual interest on the two investments is \$400. Find the two investments.

8. Part of \$8000 is invested at 4% and the rest at 5%. The 4% investment yields \$50 more interest in a year than the other. How much is invested at each rate?

9. One part of \$9000 is invested at 4% and the rest at 5%. If the two incomes thus yielded are equal, find each investment.

10. A certain sum invested at 6% yields the same yearly income as an investment of \$6000 at 5%. What is the sum invested?

11. Two men worked 9 days and together received \$144 in wages. If one man received \$7 per day, how much did the other get?

12. A boy received $\frac{1}{2}$ as much pay as his father. If together they got \$48 for 4 days' work, what was the daily wage of each?

13. If two common laborers and four skilled workmen receive \$80 per day altogether, and if the wages for skilled labor are twice as much as for common labor, find the wage of each.

14. Ten men were employed, some at \$6 per day and some at \$8 per day. The total daily wages amounted to \$68. Find the number employed at each wage.

15. Nine men were employed, some at \$7 per day and some at \$9 per day. If the total daily wages amounted to \$75, find the number working at each wage.

16. $P = 325$, $t = 2$, $r = .06$. Find I.

17. $P = 450$, $t = 3$, $r = .05$. Find A.

18. $I = 18$, $t = 3$, $P = 100$. Find r.

19. $I = 60$, $P = 500$, $r = .06$. Find t.

20. $A = 770$, $t = 2$, $r = 5\%$. Find P and I.

21. $A = 690$, $t = 3$, $r = 5\%$. Find P and I.

22. $I = 80$, $t = 2$, $r = .04$. Find P.

23. A man has \$1.25 in nickels and dimes. If there are three times as many nickels as dimes, find the number of each.

Let $\quad x = \text{no. dimes.}$

Then $\quad 3x = \text{no. nickels.}$

$$10x = \text{the value of the dimes in cents.}$$

$$5(3x) = 15x = \text{the value of the nickels in cents.}$$

$$10x + 15x = 125.$$

Solving, we have

$$x = 5.$$

$$3x = 15.$$

24. A man has $3 in nickels and dimes, there being twice as many dimes as nickels. Find the number of each.

25. The sum of $1.90 is made up of quarters plus $\frac{2}{3}$ as many dimes. How many of each are there?

26. A boy has twice as many nickels as dimes and twice as many dimes as quarters. How many of each has he if the total sum is $1.30?

27. A man has $3.25 in nickels, dimes, and quarters. He has the same number of dimes as of nickels, and twice as many quarters as dimes. Find the number of each.

28. A man has $\frac{2}{3}$ as many quarters as nickels, and as many dimes as the total number of the other coins. How many of each has he if their total value is $2.30?

4.11. *Mixture problems*

Many problems arise from the forming of mixtures of various materials. When two or more substances are mixed, an equation may be formed on the basis of the fact that the quantity of a certain substance in one material plus the quantity of this same substance in a second material is equal to the quantity of that substance in the mixture formed by combining the two materials. This same principle will extend to any mixture containing a given number of materials.

When the value of a mixture is the matter of chief interest, as in Example 2 below, we form the equation not directly on the basis of quantity but on the principle of value. When two substances are mixed we say that the value of the first one plus the value of the second equals the value of the mixture.

Example 1. How much metal that is 5% silver must be added to 10 pounds of metal, 2% of which is silver, to form a mixture having 3% silver?

Solution.

Let $x =$ no. lbs. of metal containing 5% silver.

$$\begin{aligned} \text{Then} \qquad .05x &= \text{no. lbs. of silver in the first metal;} \\ (.02)(10) &= \text{no. lbs. of silver in the second metal;} \\ (.03)(x + 10) &= \text{no. lbs. of silver in the mixture;} \\ .05x + (.02)(10) &= (.03)(x + 10). \end{aligned}$$

Solving, we have

$$x = 5.$$

Example 2. How many pounds of coffee worth 28¢ per pound must be added to 40 pounds of coffee worth 35¢ per pound to form a mixture worth 30¢ per pound?

$$\begin{aligned} \textit{Solution.}\ \text{Let } x &= \text{no. lbs. of 28¢ coffee added.} \\ 28x &= \text{the value of the 28¢ coffee in cents.} \\ 35(40) &= \text{the value of the 40¢ coffee in cents.} \\ 30(x + 40) &= \text{the value of the mixture in cents.} \\ 28x + 35(40) &= 30(x + 40). \end{aligned}$$

Solving, we have

$$x = 100.$$

EXERCISE 21

1. How much 10% copper alloy must be added to 20 pounds of 15% copper alloy to form a 12% copper alloy?

2. How many pounds of cream containing 25% butter fat must be added to 30 pounds of milk containing 5% butter fat to form a mixture containing 15% butter fat?

3. How much 30% gold alloy must be added to 25 ounces of 20% gold alloy to produce a 27% gold alloy?

4. How much pure gold must be taken from 23 ounces of 20% gold alloy to reduce it to a 15% gold alloy?

5. How much pure copper must be taken from 35 pounds of 40% copper alloy to reduce it to a 25% copper alloy?

6. How much pure silver must be taken from 45 ounces of 35% silver alloy to reduce it to a 20% silver alloy?

7. How much candy, worth 25¢ per pound, must be mixed with 20 pounds of candy worth 13¢ per pound to form a mixture of candy worth 20¢ per pound?

8. Two kinds of candy worth 15¢ and 25¢ per pound are mixed to form 30 pounds of candy worth 18¢ per pound. How much of each is used?

9. Two kinds of coffee worth 30¢ and 22¢ per pound are mixed to form 100 pounds of 25¢ per pound coffee. How much of each is used?

10. A lady bought 10 pounds of grapes for $1.35. Some of them sold for 10¢ per pound and some for 15¢ per pound. How many pounds of each kind did she buy?

11. A dealer bought 15 cases of fruit for $41.50. Some cases cost $3.00 and some $2.50. How many cases of each did he buy?

4.12. *Lever problems*

A *lever* is a mechanical device by which a force applied at one point is transferred to a second point and there intensified. When we roll over a huge rock with a crowbar we apply this principle. Many problems in algebra are concerned with levers and related mechanical devices such as pulleys.

Some preliminary definitions are necessary.

The *fulcrum* is the non-moving support upon which the lever swings when force is applied to it.

The *moment* of any force, for a given set of units, is equal to the number of units in the force multiplied by the number of units in the distance from the fulcrum to the point on the lever where the force is applied.

Question. What is the moment of a force of 10 pounds applied 5 feet from the fulcrum of a lever?

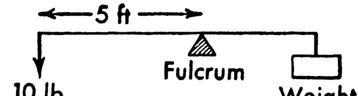

Fig. 2

Answer. Moment $= 5 \cdot 10 = 50$ foot pounds, for the given units.

The force shown in the above diagram causes the lever to rotate about the fulcrum in a counterclockwise direction

as we view the figure. Any force acting downward on the lever to the right of the fulcrum, such as the indicated weight, would tend to rotate the lever in the opposite, or clockwise, direction.

If the sum of the moments of all forces tending to rotate the lever counterclockwise is equal to the sum of the moments of all forces tending to rotate it clockwise, the lever will be balanced and stationary. In most lever problems we calculate either the force or distance necessary to make the lever balance.

Example 1. If a force of 50 pounds is applied 5 feet from the fulcrum of a lever, where must a weight of 60 pounds be placed to balance it?

Solution. (A diagram similar to Fig. 2 may be drawn.)

Let x = no. ft. from the fulcrum to the weight.

$5 \cdot 50 = 250 =$ the moment of the 50-lb. force.

$60x =$ the moment of the 60-lb. weight.

$60x = 250.$

$x = \frac{25}{6}$, so that the weight must be 4 ft., 2 in. from the fulcrum.

In the above example the weight of the lever is not considered. This weight may be important, however, in the case of a heavy lever such as a plank. To allow for it the lever is considered as composed of two parts called *arms*, each of which lies wholly on one side of the fulcrum. If the lever is uniform, or of the same weight for each unit of length, the moment of an arm may be found by multiplying its weight by one-half its length, as expressed in the units chosen for the problem. In other words, the moment of a lever arm equals the moment of a force equal to its weight applied at a point halfway from the fulcrum to the end of the arm.

Example 2. The fulcrum of a lever is 10 feet from one end and 4 feet from the other. If the beam weighs 8 pounds per

foot, what weight must be placed at the end of the shorter arm to balance the lever?

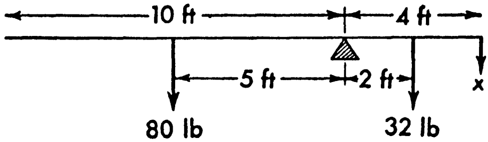

Fig. 3

Solution. The weights of the two arms are 80 and 32 pounds respectively. We must equate the counterclockwise (c.c.) and clockwise (c.) moments.

Let x = no. lbs. of force applied at the end of the shorter arm.

$80 \cdot 5$ = the moment of the left arm in the diagram (c.c.).

$32 \cdot 2$ = the moment of the right arm (c.).

$4x$ = the moment of the force x (c.).

$$4x + 32 \cdot 2 = 80 \cdot 5.$$

Solving, we have

$$x = 84.$$

If two persons are carrying a weight swung on a pole, we may consider the weight as a fulcrum and the supporting forces as the acting ones. This is then a lever just like the others we have discussed, except that it is upside down.

Example 3. Two persons carry a weight of 100 pounds swung on a 9-foot pole. Where should the weight be placed so that one person will carry 60 pounds and the other 40 pounds?

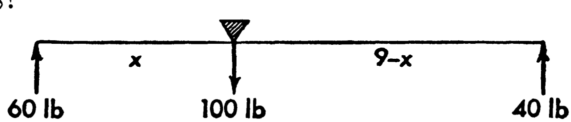

Fig. 4

Solution.

Let x = no. of ft. from weight to man carrying 60 lbs.

$9 - x$ = no. ft. from wt. to man carrying 40 lbs.

$$60x = 40(9 - x).$$

Solving, we have

$$x = \tfrac{18}{5},\ \text{or } 3\tfrac{3}{5}.$$
$$9 - x = \tfrac{27}{5},\ \text{or } 5\tfrac{2}{5}.$$

EXERCISE 22

1. A 90-pound boy and a 60-pound boy balance on a 12-foot teeter board. Where is the fulcrum?

2. What weight must be placed 6 feet from the fulcrum to balance a 70-pound weight 8 feet on the other side of the fulcrum?

3. How much weight can a 160-pound man raise with a 10-foot beam placed so that the weight is 2 feet from the fulcrum? *

4. What force is necessary to raise a weight of 1000 pounds with a 12-foot beam placed so that the weight is 3 feet from the fulcrum?

5. What weight must be placed 5 feet from the fulcrum to balance a 12-pound weight 4 feet on the other side of the fulcrum?

6. If the fulcrum is 4 feet from each end of a beam whose arms are uniform but unlike, weighing 75 and 40 pounds respectively, where should a 50-pound weight be placed to balance the beam?

7. Two men carry a 90-pound weight swung from a pole 8 feet long. Where must the weight be placed so that they will carry 40 and 50 pounds respectively?

8. A 12-foot beam weighs 7 pounds per foot. If the fulcrum is 4 feet from one end, what force must be applied at the end of the shorter arm to balance the beam?

9. A 14-foot beam weighs 8 pounds per foot. If the fulcrum is 6 feet from one end, what weight must be placed at the end of the shorter arm to balance the beam?

10. What force applied 7 feet from the fulcrum will balance 1500 pounds 2 feet from the fulcrum?

11. What force applied 8 feet from the fulcrum will balance 2000 pounds 2 feet from the fulcrum?

* In this and subsequent problems subject to two interpretations, assume that the fulcrum is between the applied forces.

12. What weight placed 3 feet from the fulcrum can be balanced by a 170-pound force applied 7 feet on the opposite side of the fulcrum?

13. What weight can a 110-pound boy balance with a 12-foot lever if the weight is 2 feet from the fulcrum?

14. Two men carry a 70-pound weight swung on a 10-foot pole and placed 4 feet from one end. How much does each carry?

15. Two men carry a 120-pound weight swung 4 feet from one end of a 9-foot pole. How much does each carry if the pole itself weighs 20 pounds?

16. Two men carry a weight of 150 pounds swung on a pole. The one who is 4 feet from the weight carries 90 pounds. How long is the pole?

17. A weight of 300 pounds rests over the end of a lever 6 inches from the fulcrum. A boy weighing 60 pounds can just lift the weight. How long is the lever?

4.13. *Work problems*

Many problems involve the rate at which an action is being performed. For example, if a man can do a piece of work in 10 days, his rate of work is to do $\frac{1}{10}$ of it per day. If he works x days, he has performed $\dfrac{x}{10}$ of the total work. The guiding principle * in problems of this type is the fact that the product of the rate of work by the number of units of time involved is equal to the fractional amount of the work done.

Example. If A can do a piece of work in 20 days and B can do the same work in 25 days, how many days will both need to do the job when working together?

* Naturally, good judgment must be used in applying the principle. For example, if one dentist can fill a tooth in one hour, it does not follow that two dentists could fill it in half an hour.

Solution.

Let x = the no. of days needed for both to do the work.

$$\frac{1}{x} = \text{the fractional part both do in one day.}$$

$$\frac{1}{20} = \text{the fractional part } A \text{ does in one day.}$$

$$\frac{1}{25} = \text{the fractional part } B \text{ does in one day.}$$

$$\frac{1}{20} + \frac{1}{25} = \frac{1}{x}.$$

Clearing fractions (multiplying both members by $100x$) we have

$$5x + 4x = 100,$$

or

$$x = \frac{100}{9} = 11\tfrac{1}{9} \text{ days.}$$

EXERCISE 23

1. A can do a piece of work in 30 days and B can do the same job in 8 days. How long will it take for both to complete the job if they work together?

2. One pipe can fill a certain tank in 25 minutes. After this pipe has been running for 10 minutes, it is shut off and a second pipe is opened. The second pipe finishes the filling in 30 minutes. How long would it have taken the second pipe to have filled the tank alone?

3. A can do a certain amount of work in $\frac{1}{4}$ the time B requires; B can do the same amount in $\frac{1}{3}$ the time C needs. The three together can do the work in 36 days. How long will it take each of them to do the work alone?

4. A large pipe fills a tank in 12 minutes and a small pipe fills it in 18 minutes. How long will it take one large and three small pipes to fill the tank if they are all opened simultaneously?

5. John started a job which ordinarily took him 5 hours, and quit after working $1\frac{1}{2}$ hours. James finished the job in 8 hours. How long would it have taken James to do the whole job by himself?

6. A can paint a house in 40 hours and B can do it in 25 hours. After A and B have been working together for 14 hours, they are joined by C and finish the job in 6 more hours. How long would it have taken C to have done the whole job alone?

7. One faucet can fill a tank in 18 minutes and a second in 42 minutes. How long will it take to fill three-quarters of the tank if both faucets are opened at the same time?

8. If it takes John a days to do a piece of work and James b days to do the same work, how long will it take for both to do the work together?

9. A and B together can do a piece of work in 5 days. If A works a times as fast as B, find the time each would require alone.

10. A list of names can be typed in 10 hours by one typist and in 15 hours by another. How long will it take them to complete the work together?

Chapter Five

FUNCTIONS AND GRAPHS

5.1. *Functions*

Perhaps the most important word in mathematics is "function." This being the case, it is certainly advisable to study the meaning of the word as long as necessary to understand it.

*A function * of x is an expression which has a special value to go with each value assigned to x.*

Examples. $2x - 1$; $x + 3$; x; $\frac{1}{x+5}$; the number of men, in the next x men you see, who will be more than six feet tall.

Why, for instance, is $2x - 1$ a function of x? The reason is that it has an associated value for each value we assign to x. For $x = 1$, it becomes $2 \cdot 1 - 1 = 1$; for $x = 2$, it becomes $2 \cdot 2 - 1 = 3$; and so on.

Similarly, examples of functions of y are $3y$, $y^3 - 7$, $\frac{1}{y}$; some functions of z are $z^2 + 1$, $2z$, $\frac{3}{z-2}$, etc.

It will be seen that any mathematical expression containing a letter is a function of that letter. Such quantities are called *mathematical functions*, and are illustrated by all but one of the examples above. The one non-mathematical function there listed is "the number of men, in the next x men you see, who will be more than six feet tall." For every value of x there will be a definite value for this quantity (which

* Some functions of x have more than one value for a given x; but we need not consider them here.

makes it a function of x); but we cannot tell in advance what that value will be. An important goal in science is to express such related quantities as mathematical functions of each other. This goal has been reached many times, as for example with regard to the initial speed of a stone thrown up vertically and the height it reaches, or the volume of a pound of specified gas and the pressure it exerts on the container. Again, accurate knowledge of the positions of the sun, earth, and moon as functions of the time has enabled astronomers to predict eclipses far in the future.

5.2. *Notation concerning functions*

A symbol which can represent any function of x is $f(x)$, read "f of x." If in a given problem $f(x)$ stands for some particular function of x, then other symbols needed to represent other particular functions of x in that problem can be written variously as $g(x)$, $h(x)$, $F(x)$ (read "g of x," "h of x," "capital F of x,") etc.

Now if, in a given problem, $f(x) = 2x^2 + 3$, then $f(0)$, read "f of zero," means "the value of $f(x)$ when $x = 0$," and equals $2 \cdot 0^2 + 3 = 3$. Similarly, $f(1) = 2 \cdot 1^2 + 3 = 5$, $f(-2) = 2(-2)^2 + 3 = 11$, and so on.

A function of two variables, say x and y, can be represented as $f(x, y)$, and is read as "f of x and y." Thus $f(1, 2)$ would read "f of 1 and 2" and would mean the value of $f(x, y)$ for $x = 1$ and $y = 2$.

Example 1. If $f(x) = 3x - 1$ and $F(x) = x^3$, evaluate $f(0) + F(-2)$.

Answer. $(3 \cdot 0 - 1) + (-2)^3 = -1 - 8 = -9$.

Example 2. If $f(x) = 4x$ and $g(y) = y + 7$, find the value of $f(2)g(3)$.

Answer. $(4 \cdot 2)(3 + 7) = 8 \cdot 10 = 80$.

Example 3. If $f(x, y) = 3x^2 - 2xy + 6$, evaluate $f(1, 3) \cdot f(0, -1)$.

Answer. $f(1, 3) = 3(1)^2 - 2(1)(3) + 6 = 3 - 6 + 6 = 3;$
$f(0, -1) = 0 - 2(0)(-1) + 6 = 6;$
$f(1, 3) \cdot f(0, -1) = 3 \cdot 6 = 18.$

EXERCISE 24

Given $f(x) = 2x^2 + 3x - 2$, *find the values of the expressions in problems* 1–10.

1. $f(1)$. **2.** $f(0)$. **3.** $f(-1)$. **4.** $f(2)$. **5.** $f(-2)$.

6. $f(-a)$. **7.** $f\left(\frac{a}{2}\right)$. **8.** $f\left(\frac{-a}{2}\right)$. **9.** $f(a + b)$. **10.** $f(a - b)$.

Given $f(x) = 3x^2 + 2$ *and* $F(x) = 2x^2 - 3$, *find the values of the expressions in problems* 11–20.

11. $f(0) - F(0)$. **12.** $f(1) + F(1)$. **13.** $f(2) - F(1)$.

14. $f(0)F(0)$. **15.** $\frac{f(2)}{F(0)}$. **16.** $\frac{f(-1)}{F(-2)}$.

17. $f(1)F(2)$. **18.** $\frac{f(0)F(0)}{f(1)}$. **19.** $\frac{f(1)F(1)}{f(0)}$. **20.** $f(a)F(0)$.

Given $F(x, y) = \frac{3y - 7x}{2y^2 + xy}$; $G(x, y) = \frac{x - 2y}{4y - 3x}$, *find the values of the expressions in problems* 21–25.

21. $F(0, -1)$. **22.** $G(0, -1)$. **23.** $F(0, -5) \cdot G(-5, 0)$.

24. $F(2, -3) + 3G(-1, -2)$. **25.** $\frac{F(-1, 4)}{G(3, -1)}$.

26. Express the area of a circle as a function of its radius, r.

27. Express the area of a triangle with a base of 10 inches as a function of its altitude, h.

28. Express the area of a triangle whose altitude is 6 feet as a function of its base, b.

29. Express the area of a square as a function of its side, s.

30. Express as a function of t the number of feet a man walks in t seconds at 5 feet per second.

31. Express as a function of t the number of feet a car goes in t seconds at sixty miles per hour.

5.3. *The coordinate system*

The rectangular coordinate system is a simple and widely used device which enables us to show, by means of a picture, or *graph,* the manner in which the value of a function of x changes along with the value of x. Because it was invented by a seventeenth-century Frenchman named *Descartes,* it is sometimes called the *cartesian coordinate* system.

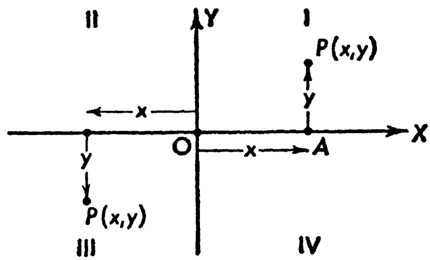

Fig. 5

The device consists of two mutually perpendicular lines, one horizontal and one vertical * (Fig. 5), drawn on a plane. The horizontal line is called the *X-axis;* the other, the *Y-axis;* and, together, they are the *coordinate axes.* They divide the plane into four parts called *quadrants,* which are numbered as shown in the figure. Their intersection, the point O, is called the *origin.*

A point such as P is located on the plane by means of two distances — OA or x and AP or y in the figure. The horizontal distance is the *abscissa,* the *x-distance,* or simply the x of P, and is positive when P is on the right of the Y-axis, zero when P is on this axis, and negative when P is to the left of it. This is conveniently indicated for memory purposes by the arrow on the right end of the X-axis, which shows the positive horizontal direction. Similarly, the vertical distance is the *ordinate, y-distance,* or y of P, and is positive, zero, or negative according as P is above, on, or below the X-axis, and also as indicated by the arrow pointing upward on the

* In this statement as well as in the subsequent discussion, we assume for simplicity that the arrow on the Y-axis points upward, as on a vertical blackboard.

Y-axis. Together, x and y are called the *coordinates* of P. It is important to note that the coordinates of P *in any quadrant* are *plus* x and *plus* y, though x stands for a negative number in quadrants II and III, as does y in quadrants III and IV.

Beginning at the origin, we mark in advance equal units on each axis on a scale to fit the problem. Then, in locating, or *plotting* a point with given numerical coordinates, designated here as x and y, we measure x units to the right or left according as the x-value is positive or negative, thus reaching a point on the x-axis. From this point we measure y units upward or downward to the designated point. Thus, to plot $A(-2, -3)$ we measure 2 units to the left of the origin and then 3 units downward. The point A, along with others, is shown in Fig. 6. Note that a capital letter before the designated coordinates, while not necessary, is convenient when the point is to be referred to again.

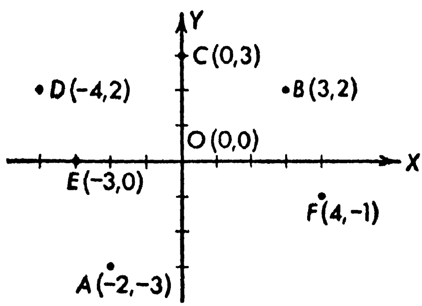

Fig. 6

5.4. *The graph of a function of x*

Using the coordinate system, let us find the geometric picture of the special function $\frac{x}{2} + 1$. For convenience we may designate this function of x by the letter y, so that

$$y = \frac{x}{2} + 1.* \tag{1}$$

* The meaning of the word *function* is here illustrated, thus: Choose any number x. Take one-half of it. Add 1 to the result. Call the sum y. Thus, the value of y depends upon the original value chosen for x.

By means of (1) we may find the value of y to pair with each of several small values of x, listing them in a brief table thus:

x	-3	-2	-1	0	1	2	3
y	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$

The points whose coordinates are listed are seen in Fig. 7 to lie in a straight line. It can be shown that all other points obtained from (1) lie also on this line. The line is said to be the *graph* of the function $\frac{x}{2} + 1$, and also the graph or *locus* of equation (1) as well as of

$$(2) \qquad 2y = x + 2,$$

obtained from (1) by clearing fractions.

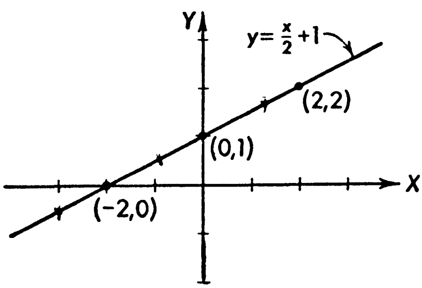

Fig. 7

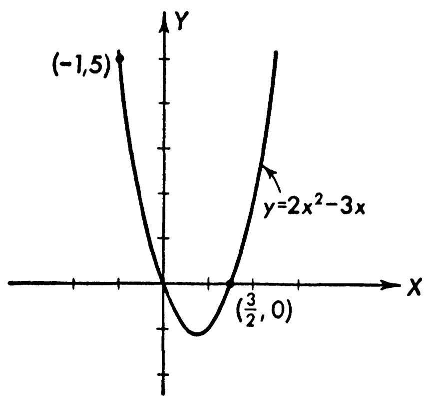

Fig. 8

Similarly, the graph of the function $2x^2 - 3x$, or of the equation

$$(3) \qquad y = 2x^2 - 3x$$

turns out to be the curve shown in Fig. 8.

The coordinates of the points plotted are shown in the following table.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
y	5	2	0	-1	-1	0	2	5

EXERCISE 25

Plot the points having the following pairs of coordinates. (Remember that the first number is the value of x.)

1. (3, 1). **2.** (1, 4). **3.** (2, 3). **4.** (3, 5). **5.** (2, 6).
6. (0, 0). **7.** (−2, 1). **8.** (3, −4). **9.** (−1, −2).
10. (0, 3). **11.** (−2, 0). **12.** (0, −1). **13.** (4, 0).

Make a table of coordinates and plot the graph of each of the following functions. (Hint: Let y equal the function.)

14. $x + 1$. **15.** $x - 2$. **16.** $x + 3$.
17. $x - 4$. **18.** $2x + 1$. **19.** $1 - 2x$.
20. $3x - 2$. **21.** $2 - 3x$. **22.** $x - \frac{1}{2}$.
23. $x + \frac{3}{4}$. **24.** $\frac{x}{2} - 1$. **25.** $\frac{x}{3} + 1$.
26. x^2. **27.** $x^2 + 2$. **28.** $x^2 - 2$.
29. $2 - x^2$. **30.** $1 - 2x^2$. **31.** $x^2 + x - 1$.
32. $2x^2 - 3x$. **33.** $x^2 - 4x - 1$. **34.** $x^2 + 2x - 2$.
35. $\frac{1}{x}$. **36.** $\frac{1}{x + 1}$. **37.** $\frac{1}{x} + 1$.
38. $\frac{1}{x^2}$. **39.** $\frac{2}{x - 1}$. **40.** $\frac{2}{x} - 1$.

5.5. *Variables and constants*

When a letter may take on an unlimited number of values in a problem, as x or y may do in equations (1), (2), and (3) of the preceding article, it is called a *variable*. If the equation is solved for y, thus being in the form

$$y = f(x), \tag{1}$$

x is called the *independent* variable, and y, or $f(x)$, the *dependent* variable. Similarly, in the equation

$$x = f(y), \tag{2}$$

y is the independent variable. To find corresponding pairs of values it is convenient to assign values to the independent variable.

A letter representing the same number throughout a problem is called a *constant.*

Constants and variables are usually taken respectively from the first and last parts of the alphabet. By common agreement some constants frequently represent the same number in many different problems, as in the case of π, the ratio of the circumference to the diameter of a circle.

5.6. *The graph of a linear function of x*

A *linear* function of x is a rational integral function of the form $ax + b$, where a and b are constants and $a \neq 0$.

Examples. $3x - 2,\ x,\ \dfrac{2x}{3},\ 5 - \dfrac{x}{2}.$

It is shown in the branch of mathematics called *analytic geometry* that the graph of the function $ax + b$, or of the equation

$$(1) \qquad y = ax + b,$$

is a straight line. Evidently it crosses the Y-axis at the point $(0, b)$. The second point needed to determine the line may be found by assigning any value to x at will. Thus, $y = x + 1$ goes through $(0, 1)$ and $(2, 3)$; $y = 2x$, through $(0, 0)$ and $(3, 6)$.

5.7. *The graph of a linear equation in one or two variables*

The general linear equation in x or y or both may be written in the form

$$(1) \qquad ax + by = c,$$

where a, b, and c are constants and both a and b cannot be zero.

When two equations are so related that all pairs of values which satisfy one also satisfy the other, they are said to be *dependent*. Evidently, from the definition, two dependent equations have the same graph.

Example. $x + y = 1$ and $2x + 2y = 2$ are both satisfied by $(0, 1)$, $(1, 0)$, $(2, -1)$, etc.

The operations leading to dependent equations are essentially the same as those yielding equivalent equations in one variable (Art. 4.5).

That is, terms may be transposed, or both members may be multiplied or divided by any number or constant except zero. If $b \neq 0$ in (1), the solution for y, namely

$$y = -\frac{a}{b}x + \frac{c}{b}, \tag{2}$$

involves operations of this sort, so that (2) and (1) are dependent and their graphs are the same. Since y is a linear function of x in (2), this graph is a straight line.

If $b = 0$, then $a \neq 0$ by the statement following (1), so that (1) takes the form

$$ax = c, \tag{3}$$

whose graph is the vertical line through $\left(\frac{c}{a}, 0\right)$, $\left(\frac{c}{a}, 1\right)$, $\left(\frac{c}{a}, 2\right)$, etc. For evidently the value of y is not restricted in any way by (3), while x is restricted to the one value, $\frac{c}{a}$.

Example. $2x = 3$, or $x = \frac{3}{2}$, is the equation of a line parallel to the Y-axis and $\frac{3}{2}$ units to the right.

Similarly, if $a = 0$, so that $b \neq 0$, the graph of

$$by = c \tag{4}$$

is a horizontal line.

Example. The line $y = -2$ is parallel to the X-axis and 2 units below it.

Our argument, then, leads to this conclusion:

The graph of a linear equation in x or y or both is a straight line.

EXERCISE 26

Plot the graph of each of the following equations.

1. $x - y = 0$.
2. $x + y = 0$.
3. $x + y - 1 = 0$.
4. $x - y + 1 = 0$.
5. $2x - y = 2$.
6. $3x + y - 6 = 0$.
7. $4x - y + 4 = 0$.
8. $2x - 3y + 6 = 0$.
9. $3x + 2y - 6 = 0$.
10. $2x - y + 4 = 0$.
11. $2x + y - 4 = 0$.
12. $3x + y = 5$.
13. $2x + 3 = 0$.
14. $5y - 1 = 0$.
15. $x = -1$.
16. $y = -5$.
17. $x = 0$.
18. $y = 0$.

19–30. Solve each equation in problems 1–12 for y in terms of x, and then solve it for x in terms of y.

31. Given $5F - 9C = 160$, the relation between the Fahrenheit (F) and centigrade (C) temperature readings, express: (a), F as a function of C; (b), C as a function of F; (c), $5(F + 40)$ as a function of C.

32. Given $I = Prt$, express: (a), P as a function of I, r, and t; (b), r as a function of I, P, and t; and (c), t as a function of I, P, and r.

5.8. *Graphs of non-mathematical functions* *

Often it is helpful and suggestive to make a graph showing the relation between two quantities even though it may not yet be possible to express one as a mathematical function of the other. In business, science, war, and practically every field of human activity, graphs or charts are used very extensively.

* This article may be omitted without loss of continuity.

For example, the meteorologist, or "weather-man," might record the Fahrenheit temperature in a given city for every third hour of the day, getting the following result:

Time	A.M. 0	3	6	9	P.M. 12	3	6	9	12
Temperature	4	−6	−6	5	20	35	25	14	6

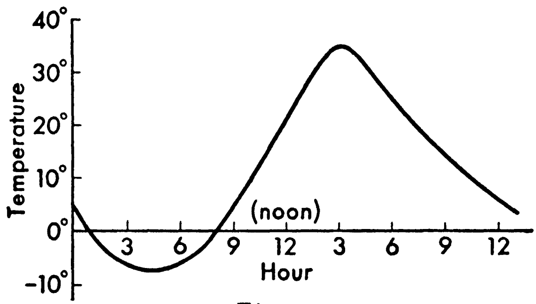

Fig. 9

The graph (Fig. 9) with the time as the independent variable shows clearly how the temperature has changed during the day. In this case it is advisable to draw a smooth curve through the plotted points to indicate that the temperature changes continuously, or from moment to moment. On the other hand, if we record quantities such as the yield per year of an acre of wheat, the smoothly curved line would be meaningless, and a broken line such as that in Fig. 10 is better.

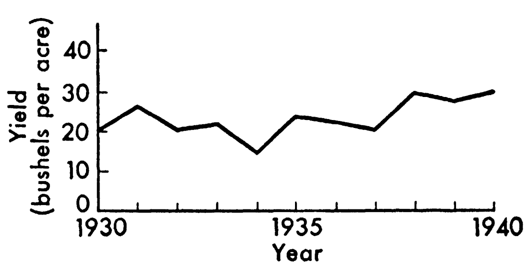

Fig. 10

The types of graphs used in practice are much too numerous to be shown here. Many of them, however, are based upon some modification of the coordinate system. Perhaps the most frequently used independent variable is *time*, which

may be measured in seconds, minutes, hours, days, years, etc. It is interesting to note that sometimes in this type of graph, the results may be carried more or less reliably into the future.

EXERCISE 27

Graph the relations indicated by the data in the following problems. Use the type of curve (smooth or broken) which seems to be more appropriate for each problem.

1. Given the following readings, graph the temperature as a function of the time. Can you suggest an explanation of the unusual conditions here indicated?

Time (A.M.)	4	5	6	7	8	9	10	11	12
Temp. (F)	60	62	65	69	75	75	65	55	55

2. Given the following data, graph the yield per year of a certain farm in bushels of corn.

Year	1920	1924	1928	1932	1936	1940	1944
Yield	600	750	600	550	510	675	790

3. Graph the indicated weight of a boy as a function of his age:

Age (years)	1	2	3	4	5	6	7	8	9	10
Weight (lbs.)	15	25	33	39	44	49	54	60	68	77

4. A river gauge registered as follows. Graph the water level as a function of the time.

Hour (A.M.)	4	5	6	7	8	9	10	11	12
Level (feet)	0	$\frac{5}{12}$	$\frac{5}{6}$	$\frac{5}{4}$	2	$\frac{3}{2}$	1	$\frac{1}{4}$	$-\frac{5}{12}$

Chapter Six

SIMULTANEOUS LINEAR EQUATIONS

6.1. *Problems leading to simultaneous equations*

Sometimes we are confronted by a situation in which two unknowns are related in two separate ways. In this case the rules of algebra enable us to find the values of the unknowns.

Example. After a storm, a mixture of water and oil filled a 100-gallon cask. The oil would eventually rise to the top, where it could be skimmed off; but it was so heavy that the process would take longer than the buyer wanted to wait. It was found that the liquid weighed 800 pounds, exclusive of the cask. It was known that the water and oil weighed about 8.4 and 7.4 pounds per gallon respectively. The deal was completed at once after a short algebraic calculation. How?

Solution. Let x and y be the number of gallons of water and oil, respectively, in the cask. We have at once, since there were 100 gallons altogether,

$$(1) \qquad x + y = 100.$$

Also, since x gallons of water and y gallons of oil weigh $8.4x$ and $7.4y$ pounds, respectively,

$$(2) \qquad 8.4x + 7.4y = 800.$$

Multiplying both sides of (1) by 7.4, we get

$$(3) \qquad 7.4x + 7.4y = 740.$$

Subtracting the two members of (3) from the corresponding ones of (2), we find that $x = 60$; and then, from (1), that $y = 100 - x = 100 - 60 = 40$. Hence there were 60 gallons of water and 40 gallons of oil in the cask.

Statements (1) and (2), which together carry in disguise the values of the unknowns x and y, are called *simultaneous equations.*

Just as an equation in one unknown may be looked upon as a sentence-question, so a set of two or more simultaneous equations may be considered as a compound question instead of a group of sentences. Taken together, they propose a question and provide the answer, in concealed form, to anyone able to find it.

6.2. *Algebraic solutions of simultaneous linear equations*

The solution of (1) and (2) in the preceding article was obtained by use of a combination of the "addition and subtraction" and "substitution" methods. We shall now illustrate these formal procedures as applied to the simultaneous equations:

(1) $$3x - 4y = 2,$$

and

(2) $$4x + 3y = 11.$$

A. *Addition and Subtraction Method*

(*Note.* The meanings of the directions in parentheses below are as here indicated: (1) $\times$ 3 means that both members of equation (1) are multiplied by 3; (3) + (4) means that corresponding members of equations (3) and (4) are added; and so on.)

(3)	(1) $\times$ 3	$9x - 12y = 6.$
(4)	(2) $\times$ 4	$16x + 12y = 44.$
(5)	(3) + (4)	$25x = 50.$

Hence

(6)		$x = 2.$
(7)	(1) $\times$ 4	$12x - 16y = 8.$
(8)	(2) $\times$ 3	$12x + 9y = 33.$
(9)	(8) $-$ (7)	$25y = 25.$

Hence

(10) $$y = 1.$$

The solution, then, is: $x = 2; y = 1.$

B. *Substitution Method*

(11) Solve (1) for y. $$y = \frac{3x - 2}{4}.$$

(12) Substitute in (2). $$4x + \frac{3(3x - 2)}{4} = 11.$$

(13) Solve (12). $$x = 2.$$

(14) Substitute in (1). $$3 \cdot 2 - 4y = 2.$$

Hence $$y = 1.$$

Solution: $x = 2,\ y = 1.$

C. *Combination Method*

The method used in solving the illustrative example in Art. 6.1 consists of solving for one unknown by method A, and then substituting this value in either equation to find the second unknown. This method is a combination of the other two.

It should be kept clearly in mind that the solution of two simultaneous equations consists of a *pair of values* which satisfies *both* equations. Applying this test to our solution of (1) and (2), we have

$$3 \cdot 2 - 4 \cdot 1 = 2 \text{ (correct)};$$

(15) $$4 \cdot 2 + 3 \cdot 1 = 11 \text{ (correct)}.$$

Hence the pair ($x = 2,\ y = 1$) is a solution. Furthermore, as we shall show in Art. 6.3, it is the only solution.

But why, it may be asked, does equation (5) give us the x-value of the equation pair? How can we add the $-12y$ of (3) to the $+12y$ of (4) when we know from the graphs of (3) and (4) that y can have *any* value in either equation, so that the two y's may be representing different numbers and hence need not be equal?

The answer may be stated thus:

In solving simultaneous equations we consider the letters not as variables but as unknown constants. When this is done, all is explained. Thus x and y are the same constants in each of equations (1) to (10), and their numerical values are exposed in equations (6) and (10).

The choice of the most efficient of the three methods for a given problem is a matter of judgment. In general, if the first unknown found is an integer or a very simple algebraic expression such as $2a$, $3b$, etc., it is quicker to find the second unknown by substitution; otherwise, by addition and subtraction. In some trivial cases, such as $2x = 3$, $4y = 7$, none of the three methods applies; but here, of course, the solution, $x = \frac{3}{2}$, $y = \frac{7}{4}$, is seen by inspection.

When some of the coefficients are literal, method A is usually preferable.

Example. Solve for x and y the following equations:

(16) $$ax + by = c,$$
(17) $$dx + ey = f.$$

Solution.

(18) (16) $\times e$ $$aex + bey = ce.$$
(19) (17) $\times b$ $$bdx + bey = bf.$$
(20) (18) $-$ (19) $$aex - bdx = ce - bf,$$

or

(21) $$(ae - bd)x = ce - bf.$$

Hence

(22) $$x = \frac{ce - bf}{ae - bd}.$$

23) (16) $\times d$ $\quad adx + bdy = cd.$

24) (17) $\times a$ $\quad adx + aey = af.$

25) (23) $-$ (24) $\quad bdy - aey = cd - af,$

r

26) $$(bd - ae)y = cd - af.$$

Ience

27) $$y = \frac{cd - af}{bd - ae}, \quad \text{or} \quad \frac{af - cd}{ae - bd}.$$

The solution is the *pair* (22) and (27). It may be checked y direct substitution in (16) and (17).

.3. *The graphical solution of simultaneous linear equations*

If we graph the lines (1) and (2), Art. 6.2, we find that hey intersect at the point (2, 1), Fig. 11. This point lies on oth lines, and hence its coordinates must satisfy both quations.

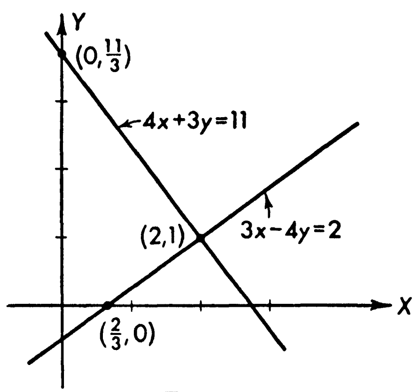

Fig. 11

In general, to solve graphically any two linear equations vith numerical coefficients, we draw the two straight lines nvolved and then find by inspection of the figure the co-rdinates of their common point, or point of intersection, f they have one. These estimated coordinates should be early the same as the correct solution-pair found alge-

braically; and of course they would be exactly the same if the figure were perfectly accurate. Thus, the graphical solution is a check on the algebraic one, and vice versa.

And now new light is thrown on algebraic difficulties which appear in some attempted solutions. For two straight lines may intersect, coincide or be parallel, and each of these three possibilities leads to a different type of algebraic result. If they intersect, they do so in a single point, and there is one and only one solution-pair. If they coincide, there are evidently infinitely many solution-pairs. Finally, if they are parallel there is no common point, and hence no solution. In these respective cases the simultaneous equations are called *independent*, *dependent*, and *inconsistent*.

To test these possibilities graphically, of course, one merely draws the lines. A simple algebraic test is the following:

Let any two linear equations in two unknowns be written so that their right members contain all terms not involving the unknowns. Then, if their left members cannot be made identical by multiplying one equation by a constant, they are independent. When the left members can be and are made identical, they are dependent when the right members are then equal, and inconsistent otherwise.

Example 1. The equations

(1) $$2x + y = 3,$$

and

(2) $$x - 3y = -2$$

are independent, since the left sides cannot be made identical. The solution-pair is (1, 1).

Example 2. The equations

(3) $$2x + y = 3,$$

and

(4) $$4x + 2y = 6$$

are dependent, since (3) can be made identical with (4) by multiplying its members by 2. Points on the common line, or simultaneous algebraic solutions, are (0, 3), (1, 1), (2, −1), etc.

Example 3. The equations

(5) $$2x + y = 3,$$

and

(6) $$4x + 2y = 7$$

are inconsistent, since when the left members are made identical the right members are unequal. Clearly no pair of values for x and y can make $4x + 2y$ equal to both 6 and 7.

EXERCISE 28

Solve by the addition and subtraction method.

1. $x + 3y = 11,$ $x - 5y = -13.$

2. $3x + 4y = -1,$ $x + 5y = 7.$

3. $11x - 6y = 4,$ $4x + 15y = 53.$

4. $9x + 7y = 8,$ $8x - 9y = -69.$

5. $9x + 8y = 3,$ $9x - 8y = -77.$

6. $7x + 3y = 2,$ $8x + 7y = -2.$

7. $9x - 5y = -1,$ $10x - 8y = 5.$

8. $5x - 11y = -4,$ $6x - 8y = -10.$

9. $7x + 3y = 4,$ $8x + 7y = 26.$

10. $2x + 6y = -3,$ $3x - 5y = 6.$

11. $15x + 4y = 7,$ $6x + 14y = 9.$

12. $3x + 5y = -9,$ $4x - 3y = 17.$

13. $7x - 2y = 15,$ $6x - y = 10.$

14. $x + 3y = 9,$ $4x + 5y = 22.$

15. $2x - 3 = 5y,$ $y + 5 = 3x.$

Solve by substitution.

16. $3x = 6,$ $2x - y = 7.$

17. $3y = 0,$ $3x + 2y = 6.$

18. $5x = -10,$ $3x + 4y = 2.$

19. $x + 4y = 16,$ $2x + 3y = 17.$

20. $7x + 6y = -11,$ $8x - 5y = -60.$

21. $6x - 9y = -7,$ $9x + 5y = 8.$

22. $10x + 12y = -7,$ $8x + 9y = -8.$

23. $2x - 3y = -29,$ $5x + 8y = 5.$

24. $15x - 14y = -15,$
$10x - 6y = -5.$

25. $3y = -9,$
$4x + 2y = 2.$

26. $4x = 12,$
$3x = y + 3.$

27. $2x + 9y = 13,$
$6x - 7y = -63.$

28–39. Graph the pairs of equations in problems 16–27, and estimate the coordinates of the points of intersection. Compare with the algebraic solutions.

Solve algebraically.

40. $\dfrac{3}{2x - y} = \dfrac{1}{2},$
$\dfrac{2}{x + y} = \dfrac{2}{3}.$

41. $\dfrac{1}{x + 2y} = \dfrac{1}{3},$
$\dfrac{2}{2x - y} = \dfrac{1}{2}.$

42. $\dfrac{5}{3x + y} = \dfrac{1}{4},$
$\dfrac{4}{x - 3y} = \dfrac{1}{3}.$

43. $\dfrac{2}{x - y} = \dfrac{1}{5},$
$\dfrac{3}{x + y} = \dfrac{1}{3}.$

44. $\dfrac{5}{3x + 2y} = \dfrac{3}{4},$
$\dfrac{3}{2x - 3y} = \dfrac{2}{3}.$

45. $\dfrac{5x}{y - 2} = \dfrac{7}{3},$
$\dfrac{y}{x - 2} = \dfrac{3}{4}.$

46. $\dfrac{3}{2} = \dfrac{5y}{2x - 3},$
$\dfrac{2}{3} = \dfrac{3x}{x + y}.$

47. $\dfrac{5}{2} = \dfrac{3x}{3 + y},$
$\dfrac{3}{4} = \dfrac{y}{3x - 2}.$

48. $\dfrac{3y}{3y + 2} = 2,$
$\dfrac{4x}{4y - 3} = 1.$

49. $ax + by = ab,$
$bx - ay = b^2.$

50. $x + y = 2a,$
$x - y = 2b.$

51. $ax + by = 2a,$
$ax - by = 2b.$

52. $ax + by = 2a^2 + ab - b^2,$
$ax - by = ab + b^2.$

53. $3x = 4,$
$5y = 7.$

54. $5ax = -10,$
$3by = 18.$

In problems 55–63, which pairs of equations have no solutions and which have more than one solution?

55. $3x - 4y = 5,$
$6x - 8y = 20.$

56. $2x - 3y = 4,$
$4x - 6y = 8.$

57. $2x = 14 - y,$
$2y = 28 - 4x.$

58. $5x = 7 - 3y,$
$6y = 10(1 - x).$

59. $6x - 5 = 10y,$
$3x - 2 = 5y.$

60. $2x = 1 + 3y,$
$6x - 9y = 3.$

61. $ax + by = 5,$
$ax + by = 10.$

62. $ax = c - by,$
$3by = 3c - 3ax.$

63. $ax - by = 0,$
$ax - by = -0.$

64. $\dfrac{1}{x} + \dfrac{3}{y} = 2,$
$\dfrac{3}{x} - \dfrac{2}{y} = \dfrac{7}{3}.$

HINT. Consider the unknowns as $\dfrac{1}{x}$ and $\dfrac{1}{y}\cdot$ The equations then become: $u + 3v = 2$ and $3u - 2v = \dfrac{7}{3}$, where $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}\cdot$ Solving, we find that $u = 1$, $v = \dfrac{1}{3}$, or $\dfrac{1}{x} = 1$, $\dfrac{1}{y} = \dfrac{1}{3}\cdot$ Hence the final solution is: $x = 1$, $y = 3$.

65. $\dfrac{2}{x} - \dfrac{1}{y} = 0,$
$\dfrac{3}{x} + \dfrac{1}{y} = \dfrac{5}{2}.$

66. $\dfrac{4}{x} + \dfrac{3}{y} = 5,$
$\dfrac{3}{x} - \dfrac{3}{y} = 2.$

67. $\dfrac{4}{x} + \dfrac{3}{y} = 5,$
$\dfrac{6}{x} + \dfrac{6}{y} = 5.$

68. $\dfrac{2}{x} + \dfrac{1}{y} = 7,$
$\dfrac{1}{x} + \dfrac{2}{y} = 8.$

69. $\dfrac{1}{x} - \dfrac{3}{y} = \dfrac{5}{2},$
$\dfrac{2}{x} - \dfrac{5}{y} = \dfrac{11}{2}.$

70. $\dfrac{2a}{x} + \dfrac{3b}{y} = 1,$
$\dfrac{2a}{x} - \dfrac{3b}{y} = 3.$

6.4. *Three equations and three unknowns*

Consider the simultaneous equations

$$(1)\qquad 2x + y + 2z = 1,$$
$$(2)\qquad 4x - y - z = 3,$$
$$(3)\qquad 6x + 2y + 3z = 2.$$

We seek now a set of three values which will satisfy (1), (2), and (3) simultaneously. Again we consider x, y, and z as constants whose values are fixed in advance, so that we may combine them at will.

Here a "guiding principle" will be found helpful to avoid "working in a circle," to wit: *Eliminate one unknown at a time, using all of the equations in which it appears*, thus reducing the problem to one with fewer unknowns and fewer equations. For example, to solve equations (1) to (3) we may eliminate z first (if we wish), thus:

(4)	(1) unchanged	$2x + y + 2z = 1.$
(5)	(2) $\times$ 2	$8x - 2y - 2z = 6.$
(6)	(4) + (5)	$10x - y = 7.$

Equation (3) must now be used, either with (1) or with (2).

(7)	(3) unchanged	$6x + 2y + 3z = 2.$
(8)	(2) $\times$ 3	$12x - 3y - 3z = 9.$
(9)	(7) + (8)	$18x - y = 11.$

There remain *two equations and two unknowns*, (6) and (9). Solving these, we get $x = \frac{1}{2}$, $y = -2$. Substitution of these values in (1) yields $z = 1$. The tentative (or untested) solution is then: $x = \frac{1}{2}$, $y = -2$, $z = 1$. The test is not complete until it is shown that these values satisfy *all three* original equations.

If a letter is missing in one of the simultaneous equations, it is usually advisable to eliminate this letter first. Thus, given

(10) $$2x - 3y - z = 1,$$

(11) $$x + 2y + 2z = 2,$$

(12) $$y - 3z = 2,$$

x should be removed from (10) and (11), and the resulting equation can then be used with (12) to get y and z.

Thus far we have considered equations with two and three unknowns. In general, if the number of simultaneous equations is the same as the number of unknowns, it is often (though not always) possible to find a set of values which will satisfy all of the equations. The guiding principle stated above equation (4) holds good in all cases.

EXERCISE 29

Solve the following systems of equations.

1. $x + y + z = 6,$
$x + y - z = 0,$
$x - y + z = 2.$

2. $x - y + z = 0,$
$x + y - z = 2,$
$x - y - z = 4.$

3. $x - y + z = 4,$
$x + y - z = 2,$
$x - y - z = -4.$

4. $x + y + z = 0,$
$x - y - z = 2,$
$x - y + z = 2.$

5. $x + 2y + z = 1,$
$2x - y - 2z = 2,$
$x - 2y + 2z = 3.$

6. $2x - 3y + z = 5,$
$x + 2y - 2z = 3,$
$3x + y - z = 2.$

7. $5x + 3y - z = 4,$
$4x - y + 2z = 3,$
$3x + 4y + z = 5.$

8. $6x - 5y + 3z = 4,$
$5x + 3y - 2z = 5,$
$3x + 2y + z = 2.$

9. $4x - 2y + 3z = 5,$
$3x + y + 2z = 3,$
$2x - 3y + 5z = 4.$

10. $2x = 3,$
$2x - y = 4,$
$3x + y - 2z = 5,$
$x + y + z + w = 6.$

11. $x + 2y = 1,$
$y + 2z = 2,$
$z + 2w = 3,$
$x + 2w = 4.$

12. $x - y + 2z = 1,$
$y + z - w = 2,$
$x - 2z + w = -1,$
$2x + y - 2w = 0.$

13. $A + 4B + C = 7,$
$A + 2B - C = 2,$
$3A - 2B + 2C = 12.$

14. $L + \frac{1}{3}M + \frac{1}{4}N = 3,$
$2L - \frac{5}{6}M + \frac{1}{2}N = \frac{3}{2},$
$\frac{3}{2}L + \frac{2}{3}M - \frac{5}{8}N = 1.$

15. $\dfrac{x+y}{3} = \dfrac{x+z}{4} = \dfrac{y+z}{5},$
$x + y + z = 6.$

16. $\dfrac{1}{x} + \dfrac{2}{y} + \dfrac{2}{z} = 1,$
$\dfrac{2}{x} + \dfrac{1}{y} - \dfrac{2}{z} = 1,$
$\dfrac{3}{x} + \dfrac{4}{y} - \dfrac{4}{z} = 2.$

6.5. *Determinants*

A *determinant* is a set of numbers or expressions arranged in rows and columns, having the same number of rows as of columns, and enclosed within two vertical bars.

Example. $\begin{vmatrix} 2 & -1 & 5 \\ 3 & 2 & -1 \\ 1 & -2 & 4 \end{vmatrix}$

The vertical bars may be considered the mark of a determinant. The numbers between them are called *elements*. The

whole expression has a definite value found by rules which will be explained below.

If the determinant has two rows and two columns, it is said to be a *second order* determinant. If there are three rows and three columns, it is of the third order. In any determinant, the order is the same as the number of rows or of columns.

The value of a second order determinant is the product of the upper left and lower right elements minus the product of the lower left and upper right elements. In other words, we find the product of the numbers read diagonally downward and subtract the product of numbers read diagonally upward — always from left to right.

Example 1. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example 2. $\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} = (3)(-1) - (1)(-2) = -3+2 = -1.$

The value of a third order determinant may be found in a simple manner when a fourth column like the first one and a fifth like the second are placed to the right of the determinant. The value is the sum of three products of elements read diagonally downward to the right minus the sum of three similar products read diagonally upward.

Example 3.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right.$$
$$= (aei + bfg + cdh) - (gec + hfa + idb).$$

By looking at the letters in the answer as read from the original determinant, the student can see easily how to evaluate the determinant *without* adding the two extra columns, if he so desires. He might try this shorter scheme in checking through the next example.

Example 4.

$$\begin{vmatrix} 3 & -1 & 2 \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = [(3)(2)(1) + (-1)(3)(2) + (2)(1)(-3)]$$
$$- [(2)(2)(2) + (-3)(3)(3) + (1)(1)(-1)]$$
$$= [6 + (-6) + (-6)] - [8 + (-27) + (-1)]$$
$$= -6 - (-20) = -6 + 20 = 14.$$

These short-cut methods of evaluating second and third order determinants do not apply to those of higher order. The method which does apply to determinants in general is called "expansion by minors." It is illustrated by the following example:

Example 5.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$
$$= aei + bfg + cdh - ahf - bdi - cge.$$

In the above expansion the second order determinants are called the *minors* of the letters before them. The minor of c, for example, is the determinant $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$, found by striking out the row and column of the original determinant in which c appears; and the other minors are found similarly. The sign in front of the letter which is multiplied by its minor in the expansion is found by starting with the upper left element and calling off the signs alternately as "plus, minus, plus, minus," each successive element reached being immediately to the right of, or below, the one just called. If an element comes on the minus count, its sign is changed; otherwise not. To illustrate, the determinant in Example 5 may be expanded by minors as above or by using the elements of any row or column as multipliers. If the second column is used, for instance, the expansion is

$$- b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}.$$

The value of the determinant is always the same, regardless of the method of expansion, as the student may verify.

EXERCISE 30

Evaluate the following determinants by the "diagonal multiplying" method.

1. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}.$ 2. $\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}.$ 3. $\begin{vmatrix} 5 & 3 \\ 2 & -2 \end{vmatrix}.$

4. $\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix}.$ 5. $\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix}.$ 6. $\begin{vmatrix} 7 & -10 \\ 2 & -3 \end{vmatrix}.$

7. $\begin{vmatrix} -5 & -3 \\ 7 & -2 \end{vmatrix}.$ 8. $\begin{vmatrix} 6 & -2 \\ -3 & 4 \end{vmatrix}.$ 9. $\begin{vmatrix} -8 & 2 \\ -8 & 2 \end{vmatrix}.$

10. $\begin{vmatrix} 3 & 6 \\ 2 & -4 \end{vmatrix}.$ 11. $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{vmatrix}.$ 12. $\begin{vmatrix} 2 & 0 & 3 \\ 1 & 3 & 2 \\ -1 & 2 & -3 \end{vmatrix}.$

13. $\begin{vmatrix} 5 & -1 & 2 \\ -3 & 2 & -1 \\ 0 & 2 & 1 \end{vmatrix}.$ 14. $\begin{vmatrix} 2 & 5 & 0 \\ -1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}.$ 15. $\begin{vmatrix} 6 & 2 & -3 \\ 0 & 1 & 2 \\ -3 & 2 & 0 \end{vmatrix}.$

16. $\begin{vmatrix} 4 & -3 & 5 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{vmatrix}.$ 17. $\begin{vmatrix} -2 & 0 & 5 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}.$ 18. $\begin{vmatrix} 4 & 1 & 2 \\ 4 & 1 & 2 \\ -3 & 1 & 5 \end{vmatrix}.$

19. $\begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 4 \\ 3 & 3 & 5 \end{vmatrix}.$ 20. $\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix}.$ 21. $\begin{vmatrix} A & D & A \\ B & E & B \\ C & F & C \end{vmatrix}.$

22–32. Check the results in problems 11–21 by the method of expansion by minors.

6.6. *Solving systems of equations by use of determinants*

The value of any unknown in a system of equations may be represented by the ratio of two determinants.

Example. Solve the system, $2x - 3 = y,$
$x + 1 = 3y.$

Solution. First, arrange the equations thus: $2x - y = 3,$
$x - 3y = -1.$

Then
$$x = \frac{\begin{vmatrix} 3 & -1 \\ -1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix}} = \frac{-10}{-5} = 2,$$

and
$$y = \frac{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix}} = \frac{-5}{-5} = 1.$$

It will be noted that the determinant in each denominator is composed of the coefficients of x and y arranged in the same order in which they occur in the equations. The determinant in each numerator is the same except that the numbers to the right of the equality signs replace the coefficients of the letter whose value is sought.

The student may verify the solutions given for the equations $\begin{pmatrix} ax + by = c \\ dx + ef = f \end{pmatrix}$ in section 6.2 C by now using the determinant method for finding the values of x and y. The latter method gives

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{ce - bf}{ae - bd}; \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{af - dc}{ae - bd}.$$

It follows that this method will give the correct results for all allowable * values of the coefficients a, b, c, d, e, and f.

Systems of three equations in three unknowns may be solved in a similar way.

Example. Solve the system $3x - 2y - z = 3;$
$2x + 3y + z = 1;$
$x - y - 2z = -2.$

* Note that if $ae - bd = 0$, the values given for x and y are meaningless, since division by zero is not permissible. In this case the equations are inconsistent or dependent, and should be studied by the method explained in Art. 6.3.

Solution.

$$x = \frac{\begin{vmatrix} 3 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-18+4+1-6+3-4}{-18-2+2+3+3-8} = \frac{-20}{-20} = 1.$$

$$y = \frac{\begin{vmatrix} 3 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-6+3+4+1+6+12}{-20 \text{ (evaluated above)}} = \frac{20}{-20} = -1.$$

$$z = \frac{\begin{vmatrix} 3 & -2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & -1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{vmatrix}} = \frac{-18-2-6-9+3-8}{-20} = \frac{-40}{-20} = 2.$$

Answer: $(x = 1, y = -1, z = 2)$.

Note that the denominator in each case contains the coefficients arranged in their normal order.

The only difference between the numerator and the denominator in each fraction is that, in the numerator, the constants on the right sides of the equations replace the coefficients of the letter whose value is sought. Applying this method to the general system:

$$\begin{aligned} ax + by + cz &= k \\ dx + ey + fz &= l \\ gx + hy + mz &= n, \end{aligned}$$

it follows that

$$x = \frac{N_x}{D}; \quad y = \frac{N_y}{D}; \quad z = \frac{N_z}{D};$$

where

$$N_x = \begin{vmatrix} k & b & c \\ l & e & f \\ n & h & m \end{vmatrix}; \quad N_y = \begin{vmatrix} a & k & c \\ d & l & f \\ g & n & m \end{vmatrix}; \quad N_z = \begin{vmatrix} a & b & k \\ d & e & l \\ g & h & n \end{vmatrix};$$

and

$$D^* = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & m \end{vmatrix}.$$

EXERCISE 31

1–27. Solve problems 1–27 of Exercise 28 by use of determinants.

28–36. Solve problems 1–9 of Exercise 29 by use of determinants.

6.7. *Stated problems with more than one unknown*

Often it is more difficult to translate a problem stated in English into a set of simultaneous equations than it is to solve the equations. Therefore, a consideration of the general principles applying to all such problems should prove helpful.

The essential steps in solving a stated problem are twofold.

(1) The unknowns to be sought must be identified, designated by letters (as x, y, z, w, etc.) and described clearly.

Examples of unsatisfactory starts.

Let x = time, or length, or rate, or distance, or amount.

Examples of satisfactory starts.

Let x = no. of minutes after 3 P.M.
Let x = no. of inches in length.
Let x = no. of feet per second.
Let x = no. of miles in distance.
Let x = no. of bushels of corn.

* For the case, $D = 0$, see the discussion in the preceding footnote.

(2) If there are two unknowns, two equations must be found expressing two relations between the unknowns which should be described in the problem. If there are three unknowns, three relations must be indicated, and so on. Thus as many equations as there are unknowns will be obtained, and the problem will be "set up," or ready for the formal algebraic solution.

Example. The combined age of John, Bill, and Harry is 36 years. Two years ago Bill was three times as old as John. Eight years hence Harry will be twice as old as Bill. How old is each?

Solution. In view of the question at the end we can start confidently thus:

Let x = no. of years in John's age;
y = no. of years in Bill's age;
z = no. of years in Harry's age.

At this point we re-read the first sentence and find that it states in English one relation between the unknowns. The statement goes into algebra thus:

(1) $$x + y + z = 36.$$

The second sentence deals with ages two years ago, which evidently are represented by $x - 2$, $y - 2$, and $z - 2$. Thus

(2) $$y - 2 = 3(x - 2).$$

Eight years hence the ages will be $x + 8$, $y + 8$, and $z + 8$. According to the third sentence,

(3) $$z + 8 = 2(y + 8).$$

Solving (1), (2), and (3) simultaneously we get $x = 4$, $y = 8$, and $z = 24$ (answer).

Not every problem has a direct question to indicate the unknowns and separate sentences to describe the conditions; but these quantities must always be described in one way or another.

It is interesting to note that impossible simultaneous conditions will be revealed algebraically by inconsistent equations; while conditions which may seem to be independent, but which really amount to the same thing, lead to dependent equations with more than one solution.

EXERCISE 32

1. Three times one number minus twice another equals 4. The sum of twice the first number plus three times the second is 7. Find the numbers.

2. If 5 times one number is divided by 3 times another, the result is 1. The sum of the numbers is 8. Find the numbers.

3. One-half the sum of two numbers is 6, and 3 times their difference is 6. Find the numbers.

4. Three times one number is equal to 4 times another, and half their sum is 7. Find the numbers.

5. Find two numbers such that 3 times their difference is 6, and the sum of 4 and the first number is 3 times the second.

6. If a certain number is added to the numerator and subtracted from the denominator of $\frac{5}{7}$, the result equals 3. The sum of half this number and a second one is 7. Find the numbers.

7. The rowing rate down a certain stream is 1 m.p.h. less than twice the upstream rowing rate. The rate upstream is 3 times the rate of the current. Find the rate of the current and the rate of rowing in still water.

Let $x =$ no. m.p.h. rowed in still water.

$y =$ no. m.p.h. the current flows.

Then $x + y =$ no. m.p.h. rowed downstream.

$x - y =$ no. m.p.h. rowed upstream.

(1) $x + y = 2(x - y) - 1.$

(2) $x - y = 3y.$

The simultaneous solution of (1) and (2) yields: $x = 4$, $y = 1$ (answer).

8. The rate of rowing downstream is 2 m.p.h. less than 3 times the rate of the current. The upstream rowing rate is $\frac{1}{3}$ the rate of the current. Find the rate of the current and the rate of rowing in still water.

9. A man can row downstream 5 miles in the same time it would take him to row 1 mile upstream. The rate of rowing in still water is 1 m.p.h. more than the rate of the current. Find the two last-mentioned quantities.

10. The rate of a boat in still water is 3 times the rate of the current and also 2 m.p.h. more than this rate. Find the two rates.

11. A plane can travel 350 m.p.h. with the wind, and its speed against the wind is 320 m.p.h. Find its speed in still air and the velocity of the wind.

12. A plane goes twice as fast with the wind as against it. If the velocity of the wind were doubled, the speed of the plane when flying with the wind would be 100 m.p.h. more than 3 times its speed against the wind. Find the speed of the plane in still air and the velocity of the wind.

13. The sum of the digits of a 3-digit number is 6. The units' digit is twice the hundreds' digit. If the order of the digits is reversed, the new number is 99 more than the original one. Find the number.

Let $h =$ the hundreds' digit;
$t =$ the tens' digit;
$u =$ the units' digit.

Then, $100h + 10t + u =$ the number.

(3) $h + t + u = 6.$

(4) $u = 2h.$

(5) $100u + 10t + h = 100h + 10t + u + 99.$

Solving (3), (4), and (5) simultaneously, we get $h = 1$, $t = 3$, and $u = 2$, so that the number is 132 (answer).

14. The sum of the digits of a 2-digit number is 7. If the order of the digits is reversed, the resulting number is 2 more than twice the original number. Find the number.

15. The sum of the digits of a 3-digit number is 12. The tens' digit is 1 less, and the units' digit 1 more, than the hundreds' digit. Find the number.

16. The sum of the digits of a 3-digit number is 14. The units' digit is 1 more than twice the hundreds' digit and 1 less than twice the tens' digit. Find the number.

17. If the order of the digits of a certain 3-digit number is reversed, the new number exceeds the old one by 396. The tens' digit is 3 times the hundreds' digit, and the sum of the digits is 1 more than twice the tens' digit. Find the number.

18. The length of a rectangle in inches is 1 less than twice its width. If its length is decreased by 1 inch and its width is increased by 3 inches, it becomes a square. Find its dimensions.

19. The length of a rectangle in inches is 2 less than twice its width. If its length is decreased by 2 inches and its width is increased by 3 inches, it becomes a square. Find its dimensions.

20. If the width of a rectangle is increased by 2 inches, its area is increased by 14 square inches. If its length is decreased by 2 inches, its area is decreased by 8 square inches. Find its dimensions.

21. If the length of a rectangle is decreased by 3 feet, and its width is increased by 2 feet, its area is increased by 2 square feet. If its length is decreased by 2 feet and its width is increased by 1 foot, its area is unchanged. Find its dimensions.

22. If the width of a rectangle is increased by 3 inches and its length is decreased by 4 inches, its area is unchanged and it becomes a square. Find its dimensions.

23. If $6000 is invested, part at 5% and part at 6%, the total yearly interest is $340. Find the amount invested at each rate.

24. If $10,000 is invested, part at 5% and part at 6%, and if the total yearly interest is $540, find the amount invested at each rate.

25. If $8000 is invested, part at 4% and part at 5%, and if the interest on the 4% investment is $50 more than the other, find the amount invested at each rate.

26. The interest received yearly on a 6% investment is twice as much as the interest on a 4% investment. If the total yearly interest is $360, find the amount of each investment.

27. If *A* works 3 days and *B* works 5 days, their combined pay is $60. If *A* works 5 days and *B* works 3 days, their combined pay is $68. Find the daily wage of each.

28. A man has \$2 in dimes and nickels. If he has twice as many dimes and half as many nickels as he now has, he would have \$2.50. How many of each has he?

29. A man has \$3.50 in nickels, dimes, and quarters. If he had the same number of dimes but half as many nickels and twice as many quarters, he would have \$4.50. If he traded his quarters for dimes and his dimes for quarters, he would have \$4.10. How many coins of each kind does he have?

30. If 100 pounds of bananas of two grades, one selling for 5¢ per pound and the other for 6¢, bring \$5.30, how many pounds of each grade are there?

31. One alloy is 5% pure silver and another 15% pure silver. How many pounds of each must be mixed to form 100 pounds of alloy of which 8% is pure silver?

32. A lady buys 10 pounds of grapes, of which part cost 25¢ for 2 pounds and the rest 25¢ for 3 pounds. If she paid \$1.00 altogether, how many pounds of each kind did she buy?

33. How many pounds each of 20¢ and 35¢ coffee must be mixed to make 100 pounds of coffee worth 25¢ per pound?

34. If corn meal is worth 3¢ per pound and flour 4¢ per pound, how many pounds of each does one buy to get 50 pounds for \$1.80?

Chapter Seven

EXPONENTS AND RADICALS

7.1. *The laws of exponents*

As we have already seen, the symbol a^m, when m is a positive integer, means "the base a taken m times as a factor." Thus, $2^3 = 2 \cdot 2 \cdot 2 = 8$. This process of raising to powers is called *involution.*

We shall now examine the laws concerning operations with positive integral exponents.

Perhaps even more useful than knowledge of the laws themselves is the realization of how very easily these working rules can be discovered at first hand by anyone who understands the meaning of the symbol a^m. There is not a better place to apply the arithmetic tests which are so useful in algebra. For example, which is correct: $a^m a^n = a^{m+n}$, or $a^m a^n = a^{mn}$? Try it with small numbers, say $a = 2$, $m = 2$, $n = 3$. Then, $a^m a^n = 2^2 2^3 = (2 \cdot 2)(2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$. Evidently the exponents here are *added,* and not multiplied. This suggests that the correct law is: $a^m a^n = a^{m+n}$.

Once the simplicity of the testing-by-arithmetic method is grasped, the student will be self-reliant when his memory fails. But he can save time by learning the five all-sufficient laws below, which, when supplemented by certain definitions, will be shown to hold *even when the exponents are not positive integers.* It is helpful to learn them in groups of 2, 2, and 1, called respectively the *repeated base,* the *repeated exponent,* and the *single base* cases. Easy extensions of the laws are indicated by the illustrative examples.

Repeated base cases

LAW 1. $$a^m a^n = a^{m+n}.$$

Examples. $2^3 2^4 = 2^7$; $3^4 3^2 3 = 3^{4+2+1} = 3^7$.

LAW 2. $$\frac{a^m}{a^n} = a^{m-n}.$$

Example. $\frac{2^7}{2^3} = 2^{7-3} = 2^4$.

Thus, if the same number or letter appears as a *base* in each of two exponential numbers which are multiplied or divided, *this same base* appears in the result.*

Repeated exponent cases

LAW 3. $$a^m b^m = (ab)^m.$$

Examples.
$$3^3 5^3 = (3 \cdot 5)^3 = (15)^3;$$
$$2^2 3^2 5^2 a^2 = (2 \cdot 3 \cdot 5a)^2 = (30a)^2.$$

LAW 4. $$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m.$$

Examples.
$$\frac{8^2}{2^2} = \left(\frac{8}{2}\right)^2 = 4^2;$$
$$\frac{(3a)^m b^m}{a^m} = \left(\frac{3ab}{a}\right)^m = (3b)^m.$$

That is, if the same number or letter appears as an *exponent* in each of two exponential numbers which are multiplied or divided, *this same exponent* appears in the result.

Single base case

LAW 5. $$(a^m)^n = a^{mn}.$$

Examples. $(4^3)^2 = 4^{3 \cdot 2} = 4^6$; $(2a^2b^3c^4)^3 = 2^3 a^6 b^9 c^{12}$.

Sometimes it is convenient to use the laws in reverse order, or as read from right to left. The memorization in the form

* An apparent exception to this statement appears in the example: $\frac{a^m}{a^m} = 1$. However, 1 can be written in the form a^0 by definition of a^0 (Art. 7.5). Note that Law 2 gives: $\frac{a^m}{a^m} = a^{m-m} = a^0$ (or 1).

given, however, is simpler and more likely to prevent certain common errors.

Example. Can $2^5 3^2$ be simplified by use of a law of exponents?

Answer. No, since neither the base nor the exponent is repeated.

While the definitions given later (Art. 7.5) enable us to say that the above five laws are true for *all* values of the exponents, whether positive, negative, zero, or fractional, the proofs below apply only when the exponents involved are positive integers, and when $m > n$ in Law 2.

Proof of Law 1. $a^m a^n = aaaa \cdots (m \text{ factors})\ aaa \cdots (n \text{ factors}) = aaa \cdots (m + n \text{ factors}) = a^{m+n}$.

Proof of Law 2. (assuming that $m > n$):

$$\frac{a^m}{a^n} = \frac{aa \cdots (m \text{ factors})}{aa \cdots (n \text{ factors})} = aa \cdots (m - n \text{ factors}) = a^{m-n}.$$

Proof of Law 3.

$$\begin{aligned} a^m b^m &= [aa \cdots (m \text{ factors})][(bb \cdots (m \text{ factors})] \\ &= [(ab)(ab) \cdots (m \text{ factors})] \end{aligned}$$

(by the commutative law of multiplication) $= (ab)^m$.

Proof of Law 4.

$$\frac{a^m}{b^m} = \frac{aa \cdots (m \text{ factors})}{bb \cdots (m \text{ factors})} = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \cdots (m \text{ factors})$$

(by Rule 1, Art. 3.5) $= \left(\frac{a}{b}\right)^m$.

Proof of Law 5.

$$(a^m)^n = a^m a^m \cdots (n \text{ factors}) = a^{m+m+\cdots\ (n \text{ terms})}$$

(by Law 1) $= a^{mn}$.

7.2. *Powers of a negative number*

A *power of x* is a number of the form x^n, where n is an integer. It is an even power when n is even and an odd power when n is odd.

Since $(-3)^n$, for example, has n negative factors, it is positive or negative according as n is even or odd. More generally:

For negative numbers, even powers are positive and odd powers are negative.

Thus,
$$(-2)^2 = (-2)(-2) = 4;$$
$$(-2)^3 = (-2)(-2)(-2) = -8.$$

EXERCISE 33 (ORAL)

Apply the laws of exponents in the following exercises.

1. x^2x^3.	**2.** a^5a^2.	**3.** a^4a^3.	**4.** x^7x.
5. x^6x^4.	**6.** xx^2x^3.	**7.** x^4x^2x.	**8.** x^5xx^4.
9. a^5a^2a.	**10.** $x^{10}x^2x^4$.	**11.** $(2^2)^3$.	**12.** $(a^2)^2$.
13. $(a^3)^2$.	**14.** $(x^4)^2$.	**15.** $(x^5)^3$.	**16.** $(x^2)^a$.
17. $(x^a)^3$.	**18.** $(x^2y)^2$.	**19.** $(x^2y^2)^3$.	**20.** $(x^5y^2)^4$.
21. $(xy^2)^3$.	**22.** $(x^ay^b)^2$.	**23.** $(x^2y^3)^{3a}$.	**24.** $(x^2y^{3b})^c$.
25. a^3b^3.	**26.** $a^xb^{2x}c^{3x}$.	**27.** $(x^2y^3)^6$.	**28.** $(x^4y^2)^4$.
29. 2^4a^4.	**30.** 2^5x^5.	**31.** $(-a)^3$.	**32.** $(-2a)^4$.
33. $(-.1a)^2$.	**34.** $(-.2x)^3$.	**35.** $(-abc)^3$.	**36.** $(-xy^2z^3)^4$.
37. $\dfrac{x^3}{x^2}$.	**38.** $\dfrac{a^5}{a^2}$.	**39.** $\dfrac{x^5}{x^3}$.	**40.** $\dfrac{x^{10}}{x^5}$.
41. $\dfrac{a^3x^5}{ax^3}$.	**42.** $\dfrac{x^4y^2}{x^2y}$.	**43.** $\dfrac{x^5y^6}{x^3y^4}$.	**44.** $\dfrac{x^2}{y^2}$.
45. $\left(\dfrac{-x^2}{y}\right)^2$.	**46.** $\left(\dfrac{-x^4}{y^4}\right)^3$.	**47.** $\left(\dfrac{-x^3}{y^2}\right)^4$.	**48.** $\left(\dfrac{x^5y^4}{x^3y}\right)^5$.

7.3. *Radicals, roots, and principal roots*

The *radical* $\sqrt[r]{a}$, read "the rth root of a," is a number whose rth power is a. That is, $(\sqrt[r]{a})^r = a$.

Thus, $(\sqrt[3]{8})^3 = 8$; $(\sqrt[4]{10})^4 = 10$; $(\sqrt[5]{-17})^5 = -17$.

The symbol $\sqrt{\ }$ is called the *radical sign;* the quantity below it, or a in the case of $\sqrt{a}$, is the *radicand;* and the integer r is the *index* of the root.

If the index is 2, it is customarily omitted, and the symbol $\sqrt{a}$ is read "the square root of a." In other roots such as $\sqrt[3]{a}$, or "the cube root of a," the index must be written.

There are *two* numbers whose squares are a given positive number. For instance, $(+2)^2 = 4$ and also $(-2)^2 = 4$. Thus, 4 has 2 square roots. However, the symbol $\sqrt{4}$ stands *only* for $+2$, which is called the *principal* square root of 4. The root -2 is designated as $-\sqrt{4}$.

By more advanced methods it can be shown that any number has 3 cube roots, 4 fourth roots, 5 fifth roots, etc. The symbol $\sqrt[r]{a}$, however, represents *only one* of the r rth roots of a—namely, the one which is called the *principal* rth root of a. When a is positive this root is positive; when a is negative and r is odd, it is negative.

Thus, $\sqrt{9} = 3$; $\sqrt[3]{8} = 2$; $\sqrt[4]{16} = 2$; and $\sqrt[3]{-8} = -2$, since $(-2)^3 = -8$.

In Art. 8.1 we shall consider the case in which a is negative and r is even.

7.4. *Rational and irrational numbers*

From the definition in Art. 7.3, the number $\sqrt{2}$ must be a positive number such that $(\sqrt{2})^2 = 2$. But what is it exactly? It is more than 1.4 and less than 1.5, since $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$. That is, $\sqrt{2}$ is between 1.4 and 1.5, or $1.4 < \sqrt{2} < 1.5$. Similarly we can show that $1.4142 < \sqrt{2} < 1.4143$. If $\sqrt{2}$ were *exactly* equal to, say, the decimal number 1.4142, it would then equal $\frac{14142}{10000}$ or $\frac{7071}{5000}$, and hence it would be the quotient of two integers. It can be proved * that $\sqrt{2}$ cannot be such a quotient. The same

* Assume that $\sqrt{2} = \dfrac{p}{q}$, where p and q are integers and the fraction $\dfrac{p}{q}$ is reduced to lowest terms. Then p and q have no factors in common, and q cannot be 1, since $\sqrt{2}$ is not an integer. Squaring both sides of the equation, we have: $2 = \dfrac{p^2}{q^2} = \dfrac{pp}{qq}$, which is impossible since there are no common factors in p and q to be canceled out. Hence our assumption is impossible. Similarly we can prove that $\sqrt[3]{4}$, $\sqrt[4]{7}$, etc., cannot be expressed as quotients of integers.

is true of $\sqrt[3]{4}$, $\sqrt[4]{6}$, and in general the *rth root of a number which is not a perfect rth power*. Such roots are called *surds*, and are included in the large group of *irrational numbers*.

An irrational number is one which cannot be expressed exactly as the quotient of two integers.

Examples. All surds, such as $\sqrt{2}$ and $\sqrt[3]{5}$, and many other numbers such as π, the ratio of the circumference to the diameter of a circle.

A rational number is one which can be expressed as the quotient of two integers.

Examples. $\frac{2}{3}$, $\frac{9}{5}$, $\sqrt{\frac{9}{4}}$ (or $\frac{3}{2}$), 1.41 (or $\frac{141}{100}$), 3 (or $\frac{3}{1}$).

As suggested by the alternate forms of 3 and 1.41 in parentheses above, all integers and all decimal numbers are rational. It follows that irrational numbers cannot be expressed exactly in decimal form. Most of the numbers in various tables, such as lists of square and cube roots (Table 2 in this text) are merely *decimal approximations*, or numbers as close to the actual roots as possible with the given number of decimal places.

Irrational numbers are connected in an interesting manner with incommensurable quantities in geometry. Consider, for example, the length OP in Fig. 12. It may be computed

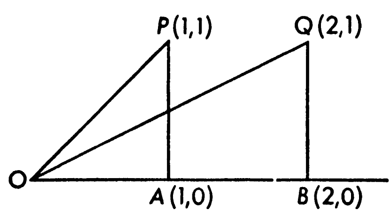

Fig. 12

by use of the famous Pythagorean Theorem, which states that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides. Since the sides of the right triangle OAP are 1 and 1, the hypotenuse OP is equal

to $\sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$. Geometrically, if OA and OP had a common unit of measure which would go into OP exactly m times, say, and into OA exactly n times, then $\frac{OP}{OA} = \frac{m}{n}\cdot$ But we know this to be impossible, since $\frac{OP}{OA} = \sqrt{2}$, and $\sqrt{2}$ cannot be expressed as the quotient of two integers. Hence OP and OA have no common unit of measure and are said to be *incommensurable*. Similarly, since OQ is $\sqrt{5}$ units long, it is incommensurable with the unit length BQ.

EXERCISE 34

1. Give the two square roots of the following numbers, naming the principal root first: (a), 1; (b), 4; (c), $\frac{1}{4}$; (d), $\frac{4}{9}$; (e), $\frac{25}{4}$; (f), x^2.

2. In each case below state whether the radical is rational or irrational. If it is rational, find its exact value; otherwise give its approximate value as found in Table 2.

(a), $\sqrt{9}$; (b), $\sqrt{10}$; (c), $\sqrt[3]{8}$; (d), $\sqrt[3]{-27}$; (e), $\sqrt[3]{9}$; (f), $\sqrt{12}$; (g), $\sqrt{16}$; (h), $\sqrt[3]{-1}$; (i), $\sqrt[3]{1}$; (j), $\sqrt[3]{-7}$; (k), $\sqrt[3]{-8}$; (l), $\sqrt{8}$.

3. Find the exact lengths of the diagonals of the rectangles whose dimensions in inches are as follows:

(a), 1×2; (b), 2×3; (c), 3×4; (d), 3×7; (e), 5×12; (f), 4×6; (g), 8×15.

4. In which cases in problem 3 are the lengths of the diagonals rational?

7.5. *Negative, zero, and fractional exponents*

It is desirable that the five laws stated in Art. 7.1 shall be true even when the exponents are not positive integers. It can be so arranged by use of the following definitions.

Definition 1. $\qquad a^0 = 1. \quad (a \neq 0.)$

By Law 1, $a^0a^m = a^{0+m} = a^m$. Also, by the definition, $a^0a^m = 1 \cdot a^m = a^m$. Or again, by division, $\frac{a^m}{a^m} = 1$; while by Law 2 and the definition, $\frac{a^m}{a^m} = a^{m-m} = a^0 = 1$. In other

words, the laws remain valid when the exponent zero is involved, given this definition.

Definition 2. $$a^{-k} = \frac{1}{a^k}.$$

Consider first a numerical example. By Law 2, $\frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$. This is correct when 2^{-3} is defined as $\frac{1}{2^3}$, since $\frac{2^2}{2^5} = \frac{2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{2^3}$.

In general, when $n > m$, $\frac{a^m}{a^n} = a^{m-n} = a^{-(n-m)}$, and by this definition, $a^{-(n-m)} = \frac{1}{a^{n-m}}$.

Two convenient rules are consequences of Definition 2.

(1) $$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

For $\left(\frac{a}{b}\right)^{-m} = \frac{1}{\left(\frac{a}{b}\right)^m} = \frac{1}{\frac{a^m}{b^m}} = \frac{b^m}{a^m} = \left(\frac{b}{a}\right)^m.$

(2) *If the term* a^{-m} *is a factor of the denominator of a fraction, it may be changed to* a^m *and written as a factor of the numerator; if it is a factor of the numerator, it may be changed to* a^m *and written as a factor of the denominator.*

Examples.

$$\frac{3^{-2}4}{5} = \frac{4}{3^2 \cdot 5} = \frac{4}{45};$$

$$\frac{a}{b^{-3}(c+d)} = \frac{ab^3}{c+d};$$

$$\frac{a}{b(c+d)^{-2}} = \frac{a(c+d)^2}{b}.$$

Note, however, that in the case: $\frac{a^{-1} - b^{-1}}{a^{-1} + b^{-1}}$, the terms with negative exponents are *not* factors of the numerator or de-

nominator. In such cases the simplification must be made by replacing a^{-1} by $\frac{1}{a}$, b^{-1} by $\frac{1}{b}$, etc., and then simplifying the complex fraction. Or, more directly,

$$\frac{a^{-1} - b^{-1}}{a^{-1} + b^{-1}} = \frac{(a^{-1} - b^{-1})ab}{(a^{-1} + b^{-1})ab} = \frac{b - a}{b + a}.$$

Definition 3. $\qquad \boldsymbol{a^{\frac{1}{r}} = \sqrt[r]{a}}.$

By Law 5, $(a^{\frac{1}{r}})^r = a^{\frac{1}{r}\cdot r} = a^1 = a$. Also, by the definition, $(a^{\frac{1}{r}})^r = (\sqrt[r]{a})^r = a$. For the special case $r = 3$, for example, $a^{\frac{1}{3}}a^{\frac{1}{3}}a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ (by Law 1) $= a^1 = a$, while $\sqrt[3]{a}\,\sqrt[3]{a}\,\sqrt[3]{a} = (\sqrt[3]{a})^3 = a$.

Definition 4. $\qquad \boldsymbol{a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{a^p}}$.*

For, by Law 5 and Definition 3, $a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p$, and also $a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = \sqrt[q]{a^p}$.

Thus, $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$, and also $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$. Here the final result is more easily calculated from the first form of Definition 4. This should be used for $a^{\frac{p}{q}}$ when a is a perfect qth power.

Again, $7^{\frac{2}{3}} = (\sqrt[3]{7})^2$, and also $7^{\frac{2}{3}} = \sqrt[3]{7^2} = \sqrt[3]{49}$. In this case the second definition of $a^{\frac{p}{q}}$ is preferable, since 7 is not a perfect cube.

Note. The reason for the use of the adjectives "rational" and "integral" as applied to *letters* is suggested in Articles 7.4 and 7.5. Just as $\sqrt{2}$ is not rational, and 2^{-1}, or $\frac{1}{2}$, is not integral, so by definition $\sqrt{x}$ is *not rational in* x, and x^{-1}, or $\frac{1}{x}$, is *not integral in* x.

* The fraction $\frac{p}{q}$ must be reduced to lowest terms. Note that $(-1)^{\frac{2}{4}}$, or $\sqrt[4]{(-1)^2}\,(= \sqrt[4]{1} = 1)$ is not the same as $(-1)^{\frac{1}{2}}\,(= \sqrt{-1})$.

EXERCISE 35

Evaluate each of the following.

1. $4^{\frac{1}{2}}$.
2. $8^{\frac{1}{3}}$.
3. $9^{\frac{3}{2}}$.
4. $4^{\frac{5}{2}}$.
5. $(16)^{\frac{3}{4}}$.
6. $(25)^{\frac{3}{2}}$.
7. $-(36)^{\frac{1}{2}}$.
8. $(-64)^{\frac{2}{3}}$.
9. $(-125)^{\frac{2}{3}}$.
10. $(27)^{\frac{2}{3}}$.
11. $-(81)^{\frac{3}{4}}$.
12. $-(64)^{\frac{2}{3}}$.
13. $(-8)^{\frac{2}{3}}$.
14. $-8^{\frac{2}{3}}$.
15. $-(125)^{\frac{2}{3}}$.
16. $(81)^{\frac{1}{4}}$.
17. $8^{\frac{4}{3}}$.
18. $(32)^{\frac{3}{5}}$.
19. $(-32)^{\frac{3}{5}}$.
20. $(-32)^{\frac{2}{5}}$.
21. $-(32)^{\frac{2}{5}}$.
22. $(243)^{\frac{2}{5}}$.
23. $(-243)^{\frac{2}{5}}$.
24. $(-243)^{\frac{3}{5}}$.
25. $2^0 \cdot 3^{-1}$.
26. $2^2 \cdot 3^0$.
27. $4^{\frac{1}{2}} \cdot 2^{-1}$.
28. $8^{-\frac{1}{3}} \cdot 2^0$.
29. $9^{\frac{1}{2}}(-3)^0$.
30. $27^{-\frac{1}{3}}(\frac{2}{3})^0$.
31. $5^0 \cdot 8^{-\frac{2}{3}}$.
32. $2^{-2} \cdot 4^{\frac{3}{2}}$.
33. $9^{-\frac{1}{2}} \cdot 8^{-\frac{2}{3}}$.
34. $2^{-2} \cdot 9^{\frac{1}{2}}$.
35. $2^{-3} \cdot 9^{\frac{3}{2}}$.
36. $(-3)^{-3}$.
37. $\left(\dfrac{-32}{243}\right)^{-\frac{2}{5}}$.
38. $\left(\dfrac{4}{9}\right)^{-\frac{1}{2}}$.
39. $(-27)^{-\frac{4}{3}}$.
40. $-(5b)^0 + (-5b)^0$.
41. $\frac{1}{3}(3a)^0$.
42. $(-0.027)^{\frac{1}{3}}$.
43. $-(\frac{1}{4})^{-2}$.
44. -5^{-1}.
45. $(-\frac{2}{3})^{-3}$.
46. $-(-3)^{-1}$.
47. $[2 \cdot 4^0 + (16)^{-\frac{3}{4}}]^{-1}$.

Change each of the following to expressions in which the exponents are positive, and in which the operations indicated in Laws 1 to 5 have been carried out completely.

48. $\dfrac{2x^0}{y \cdot 2}$.
49. $\dfrac{3x^{-3}}{2x^0}$.
50. $\dfrac{4^0x^{-1}}{3y^{-2}}$.
51. $\dfrac{2x^{-3}y^0}{3^0y^{-2}}$.
52. $\dfrac{5^0x^{-1}}{3y^{-2}}$.
53. $\dfrac{6x^{-2}}{2^0y^{-1}}$.
54. $\dfrac{2^{-1}x^0}{3^0y^{-2}}$.
55. $\dfrac{3^{-2}x^{-1}}{2^0y^{-2}}$.
56. $\dfrac{7^0x^{-2}}{2^{-1}y^{-2}}$.
57. $\dfrac{5x^{-1}y^2}{2^0xy^{-2}}$.
58. $\dfrac{4^{-1}x^0y}{2^0x^{-1}y^{-1}}$.
59. $\dfrac{2^{-1}2x^0}{3^0x^{-1}y}$.
60. $\dfrac{(2x^0y^2)^2}{(3^0x^2y^{-1})^3}$.
61. $\dfrac{(5x^{-2}y^2)^2}{(3x^2y^{-2})^2}$.
62. $\dfrac{(4^0x^2y^{-3})^4}{(200xy)^0}$.
63. $\dfrac{(3^{-1}x^{-2}y^0)^2}{2xy}$.
64. $\dfrac{(2^{-1}x^3y^0)^3}{2x^2y^{-3}}$.
65. $\dfrac{(3^0x^{-1}y^2)^{-1}}{2x^2y^{-3}}$.
66. $\dfrac{(2x^0y^{-2})^{-2}}{3x^{-1}y}$.
67. $\dfrac{(5x^{-2}y)^3}{3x^2y^{-1}}$.
68. $\dfrac{(4^0x^{-3}y^{-1})^{-2}}{2xy^2}$.
69. $\dfrac{(3^{-2}x^{-1}y^0)^{-3}}{2x^2y}$.
70. $\dfrac{(7^0a^{-1}b^0)^{-2}}{2^{-1}a^2b^{-1}}$.
71. $\dfrac{(8^0a^{-2}b)^{-3}}{3^{-1}a^2b^0}$.

72. $(x - y)^2(x + y)^{-1}$.

73. $(a + b)^{-3}(a - b)$.

74. $(2x - y)^{-1}(x + 2y)$.

75. $(3x + y)^3(x + y)^{-2}$.

76. $(2x - 3y)^{-2}(x + y)^0$.

77. $(5x - y)^0(x - y)^{-1}$.

78. $(x - y)^{\frac{1}{2}}(x - y)^{-\frac{1}{2}}$.

79. $(x + y)^{\frac{1}{2}}(x + y)^{-\frac{1}{2}}$.

80. $(a^2 - x^2)^{\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}}$.

81. $(2x^2 + a^2)^{-\frac{1}{2}}(2x^2 + a^2)^{\frac{1}{2}}$.

82. $(a^2 + x^2)^{\frac{1}{4}}(a^2 + x^2)^{-\frac{1}{2}}$.

83. $(x^2 - a^2)^{\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{4}}$.

Express each of the following as a complex fraction and simplify to a simple fraction in lowest terms.

84. $\dfrac{2}{a^{-2} + b^{-2}}$. *Solution.* $\dfrac{2}{\dfrac{1}{a^2} + \dfrac{1}{b^2}} = \dfrac{2a^2b^2}{b^2 + a^2}$.

85. $\dfrac{ab^{-2} + a^{-2}b}{a^{-1} + b^{-1}}$.

86. $(x^{-2} - y^{-2})^{-1}$.

87. $\dfrac{a(a + b)^{-1} - b(a - b)^{-1}}{a^{-1} - b^{-1}}$.

88. $\dfrac{(a^2 + x^2)^{-2} + x^{-2}}{(a^2 + x^2)^2 + x^2}$.

89. Select appropriate multipliers for each of the problems 85, 87, and 88 and simplify without first expressing each as a complex fraction.

Reduce the following to fractions whose numerators have no fractional exponents.

90. $\dfrac{(a - x)^{\frac{1}{2}} + (a - x)^{-\frac{1}{2}}}{a - x}$. *Solution.* Multiply both numerator and denominator by $(a - x)^{\frac{1}{2}}$.

$$\frac{(a - x)^{\frac{1}{2}} + (a - x)^{-\frac{1}{2}}}{a - x} \cdot \frac{(a - x)^{\frac{1}{2}}}{(a - x)^{\frac{1}{2}}} = \frac{(a - x)^1 + (a - x)^0}{(a - x)^{\frac{3}{2}}}$$

$$= \frac{a - x + 1}{(a - x)^{\frac{3}{2}}}.$$

91. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} + (a^2 - x^2)^{-\frac{1}{2}}}{a^2 - x^2}$.

92. $\dfrac{(a^2 + x^2)^{\frac{1}{2}} + (a^2 + x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{\frac{1}{2}}}$.

93. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} - (a^2 - x^2)^{-\frac{1}{2}}}{(a^2 - x^2)^{\frac{1}{2}}}$.

94. $\dfrac{(a^2 + x^2)^{\frac{1}{2}} - (a^2 + x^2)^{-\frac{1}{2}}}{a^2 + x^2}$.

95. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} + x^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 - x^2)^{\frac{3}{2}}}$.

96. $\dfrac{(a^2 + x^2)^{\frac{1}{2}} - x^2(a^2 + x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{\frac{3}{2}}}$.

Reduce the following to fractions whose denominators have no fractional exponents.

97. $\dfrac{(a^2-x^2)^{\frac{1}{2}}}{(a^2-x^2)^{-\frac{1}{2}}-a^2(a^2-x^2)^{-\frac{1}{2}}}$.

98. $\dfrac{(a^2+x^2)^{\frac{1}{2}}}{(a^2+x^2)^{\frac{1}{2}}-a^2(a^2+x^2)^{-\frac{1}{2}}}$.

99. $\dfrac{a^2-x^2}{x^2(a^2-x^2)^{-\frac{1}{2}}-(a^2-x^2)^{-\frac{1}{2}}}$.

100. $\dfrac{a^2+x^2}{x^2(a^2+x^2)^{-\frac{1}{2}}-(a^2+x^2)^{\frac{1}{2}}}$.

101. $\dfrac{a^2-x^2}{a^2(a^2-x^2)^{-\frac{1}{2}}+(a^2-x^2)^{\frac{1}{2}}}$.

102. $\dfrac{(a^2+x^2)^{\frac{3}{2}}}{a^2(a^2+x^2)^{-\frac{1}{2}}-(a^2+x^2)^{\frac{1}{2}}}$.

7.6. *Laws concerning radicals*

Operations with radicals may be performed by changing them to exponential form and then applying the laws of exponents.

Example 1. $\dfrac{\sqrt{8}}{\sqrt{2}} = \dfrac{8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \left(\dfrac{8}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = \sqrt{4} = 2.$

Example 2. $\sqrt[3]{\sqrt{2}} = (2^{\frac{1}{2}})^{\frac{1}{3}} = 2^{\frac{1}{6}} = \sqrt[6]{2}.$

However, while this method should be held in reserve in case of doubt, it is more efficient to learn directly the more frequently used rules involving radicals. The student may prove each rule by changing the radicals to exponential form.

The laws as stated below are valid when all letters stand for positive integers. (Otherwise some exceptions are necessary.)

LAW **1.** $$\sqrt[r]{a^{kr}} = a^k.$$

Examples. $\sqrt[3]{2^{12}} = \sqrt[3]{2^{4\cdot 3}} = 2^4$; $\sqrt[5]{3^{10}} = 3^2$.

LAW **2.** $$\sqrt[r]{ab} = \sqrt[r]{a}\sqrt[r]{b}.$$

From this rule we see that *a factor of the radicand which is a perfect rth power may be taken from the radicand if its rth root is written outside the radical sign.* When all such factors have been removed the radical is said to be *reduced to lowest terms.*

Example 1. $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\,\sqrt{2} = 2\sqrt{2}.$

Example 2. $\sqrt[3]{32a^4b^6} = \sqrt[3]{2^3a^3b^6(4a)} = 2ab^2\sqrt[3]{4a}.$

Example 3. $\sqrt{4a^2 + 16b^2} = \sqrt{4(a^2 + 4b^2)} = 2\sqrt{a^2 + 4b^2}.$
(Why not $\sqrt{4a^2 + 16b^2} = \sqrt{4a^2} + \sqrt{16b^2} = 2a + 4b$?)

This common error should be carefully noted and avoided.

LAW **3.**
$$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \sqrt[r]{\frac{a}{b}}.$$

Example 1. $\dfrac{\sqrt[3]{16a^7}}{\sqrt[3]{2a}} = \sqrt[3]{\dfrac{16a^7}{2a}} = \sqrt[3]{8a^6} = 2a^2.$

Example 2.
$$\frac{\sqrt{8} + 3\sqrt{20}}{2\sqrt{2}} = \frac{\sqrt{8}}{2\sqrt{2}} + \frac{3\sqrt{20}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{8}{2}} + \frac{3}{2}\sqrt{\frac{20}{2}}$$
$$= \tfrac{1}{2}\sqrt{4} + \frac{3\sqrt{10}}{2} = 1 + \frac{3\sqrt{10}}{2}.$$

LAW **4.**
$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}.$$

Examples. $\sqrt[3]{\sqrt{2}} = \sqrt[6]{2};\ \sqrt[3]{\sqrt{8}} = \sqrt{\sqrt[3]{8}} = \sqrt{2}.$

7.7. *Addition and subtraction of radicals*

Radicals are of the same *order* if the indices of the roots are equal. Thus cube roots are all of the same order, having the index 3.

Radicals of the same order whose radicands are equal are called *like* radicals. *Only like radicals can be combined by addition and subtraction.* To determine whether two radicals are like radicals, the radicands should first be reduced to lowest terms, as in the examples below.

* Historically, the symbol $\sqrt{\ }$ is a union of two symbols: $\surd$ (take the root of) and the vinculum — (treat as a single number). Thus, $\sqrt{64 + 36} = \surd\overline{64 + 36} = \sqrt{100} = 10.$

Example 1. $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$.

Example 2. $\sqrt[3]{54a} - \sqrt[3]{16a} = 3\sqrt[3]{2a} - 2\sqrt[3]{2a} = \sqrt[3]{2a}$.

The sums or differences of unlike radicals may either be left uncombined or else may be approximated in decimal form.

For example, $\sqrt{3} + \sqrt{2}$ cannot be further simplified in exact form, but $\sqrt{3} + \sqrt{2} = 1.732 + 1.414 = 3.146$ (nearly). Note that $\sqrt{3} + \sqrt{2}$ *is not* $\sqrt{5}$, since $\sqrt{5} = 2.236$ (nearly).

7.8. *Multiplication of radicals*

By the use of Law 2, the product of two radicals of the same order can be found directly. The law can be extended to handle cases where the multiplicand or multiplier or both are indicated sums of radicals.

The following illustrative examples show the direct use of Law 2:

Example 1. Find the product of $\sqrt{28}$ and $\sqrt{\frac{3}{7}}$.

Solution. $\sqrt{28} \cdot \sqrt{\frac{3}{7}} = \sqrt{(28)(\frac{3}{7})} = \sqrt{12} = 2\sqrt{3}$.

Example 2. Find the product of $\sqrt{5}+3\sqrt{2}$ and $2\sqrt{5}-4\sqrt{2}$.

Solution.

$$\begin{array}{l}
\quad\sqrt{5} + 3\sqrt{2} \\
\underline{\quad 2\sqrt{5} - 4\sqrt{2}\qquad\qquad\quad} \\
\quad 2\cdot 5 + 6\sqrt{10} \\
\underline{\qquad\qquad - 4\sqrt{10} - 12\cdot 2\quad} \\
10 + 2\sqrt{10} - 24 = -14 + 2\sqrt{10}
\end{array}$$

Thus, $(\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 4\sqrt{2}) = -14 + 2\sqrt{10}$.

EXERCISE 36

Reduce the radicands to lowest terms.

1. $\sqrt{9}$.	**2.** $\sqrt{16}$.	**3.** $\sqrt{36}$.	**4.** $\sqrt[3]{8}$.
5. $\sqrt[3]{27}$.	**6.** $\sqrt[3]{125}$.	**7.** $\sqrt{12}$.	**8.** $\sqrt{20}$.

9. $\sqrt{28}$. 10. $\sqrt{32}$. 11. $\sqrt{18}$. 12. $\sqrt{27}$.
13. $\sqrt{45}$. 14. $\sqrt{54}$. 15. $\sqrt{63}$. 16. $\sqrt{72}$.
17. $\sqrt{90}$. 18. $\sqrt[3]{-81}$. 19. $-\sqrt[3]{128}$. 20. $-\sqrt[3]{-135}$.
21. $\sqrt{96}$. 22. $\sqrt{128x^3}$. 23. $\sqrt{125x^2}$. 24. $\sqrt{50x^3}$.
25. $\sqrt{75x^2}$. 26. $\sqrt{150x}$. 27. $\sqrt{175x^4}$. 28. $\sqrt{200x^5}$.
29. $\sqrt{250x}$. 30. $\sqrt{108x^4}$. 31. $\sqrt{72x^6}$. 32. $\sqrt{180x^{13}}$.
33. $\sqrt{216x^{16}}$ 34. $\sqrt{98x^8}$. 35. $\sqrt{147x}$. 36. $\sqrt{245x^7}$.
37. $\sqrt[3]{16x^4}$. 38. $\sqrt[3]{24x^3}$. 39. $\sqrt[3]{32x^4}$. 40. $\sqrt[3]{40x^5}$.
41. $\sqrt[3]{48x^2}$. 42. $\sqrt[3]{56y^3}$. 43. $\sqrt[3]{72y^7}$. 44. $\sqrt[3]{54b^6}$.
45. $\sqrt[3]{80a^8}$. 46. $\sqrt[3]{81y^4}$. 47. $\sqrt[3]{108y^7}$. 48. $\sqrt[3]{250x^5}$.
49. $\sqrt[3]{375y^9}$. 50. $\sqrt[4]{32x^{11}}$. 51. $\sqrt[4]{48y^7}$. 52. $\sqrt[5]{32x^6}$.
53. $\sqrt[5]{64x^4}$. 54. $\sqrt[5]{96x^6}$. 55. $(\sqrt{8})(\sqrt{12})$.

Solution for problem 55. $\sqrt{8}\,\sqrt{12} = (2\sqrt{2})(2\sqrt{3}) = 4\sqrt{6}$.

56. $\sqrt{3}\,\sqrt{27}$. 57. $\sqrt{8}\,\sqrt{20}$. 58. $\sqrt{5}\,\sqrt{20}$. 59. $\sqrt{18}\,\sqrt{27}$.
60. $\sqrt{7}\,\sqrt{63}$. 61. $\sqrt[3]{8}\,\sqrt[3]{16}$. 62. $\sqrt[3]{5}\,\sqrt[3]{200}$. 63. $\sqrt[3]{16}\,\sqrt[3]{54}$.
64. $\sqrt[3]{9}\,\sqrt[3]{24}$. 65. $\sqrt[3]{16}\,\sqrt[3]{32}$. 66. $\sqrt[3]{25}\,\sqrt[3]{40}$.

67. $\dfrac{\sqrt[3]{250}}{\sqrt[3]{2}} = \sqrt[3]{\dfrac{250}{2}} = \sqrt[3]{125} = 5$. 68. $\dfrac{\sqrt{98}}{\sqrt{2}}$. 69. $\dfrac{\sqrt{48x^2}}{\sqrt{3}}$.

70. $\dfrac{\sqrt{96x^3}}{\sqrt{6}}$. 71. $\dfrac{\sqrt[3]{81x^4}}{\sqrt[3]{3}}$. 72. $\dfrac{\sqrt[3]{108x^5}}{\sqrt[3]{4}}$.

73. $\sqrt{4-8x^2} = \sqrt{4(1-2x^2)} = 2\sqrt{1-2x^2}$.
74. $\sqrt{4+12x^2}$. 75. $\sqrt{9-18y^2}$. 76. $\sqrt{9+27x^2}$.
77. $\sqrt{4-16x^2}$. 78. $\sqrt{16-4a^2}$. 79. $\sqrt{25+100x^2}$.
80. $\sqrt{4a^2-36x^2}$. 81. $\sqrt{9x^2+36y^2}$. 82. $\sqrt{25y^2+100x^2}$.

83. $\sqrt{\dfrac{1}{2}+\dfrac{1}{2}}$. 84. $\sqrt{\dfrac{4}{x^2}-\dfrac{4}{y^2}}$.

85. $\sqrt[3]{\sqrt{8}} = \sqrt{\sqrt[3]{8}} = \sqrt{2}$. 86. $\sqrt[3]{\sqrt{27}}$.
87. $\sqrt{\sqrt[3]{16}}$. 88. $\sqrt[3]{\sqrt{64}}$.
89. $\sqrt{\sqrt[3]{64}}$. 90. $\sqrt[5]{\sqrt{32}}$.
91. $\sqrt{\sqrt[3]{36}}$. 92. $\sqrt[3]{\sqrt{125}}$.

Combine the following radicals as indicated.

93. $\sqrt{27} - \sqrt{12}$.

94. $\sqrt{75} - \sqrt{12}$.

95. $\sqrt{32} - \sqrt{18}$.

96. $\sqrt{50} - 2\sqrt{8}$.

97. $\sqrt{75} + 2\sqrt{12}$.

98. $\sqrt{27a^3} + \sqrt{12ab^2}$.

99. $(1 - \sqrt{8}) + (2 + \sqrt{18})$.

100. $\sqrt{12x^3y} - \sqrt{27xy^3}$.

101. $(1 + \sqrt{8}) - (2 + \sqrt{18})$.

102. $\sqrt{98} + \sqrt{72}$.

103. $(5 - \sqrt{50}) - (4 - \sqrt{32})$.

104. $\sqrt{125x} - \sqrt{45x}$.

105. $(4 - \sqrt{75}) - (2 - \sqrt{108})$.

106. $2(\sqrt{3} + 1) - 3\sqrt{3}$.

107. $\sqrt{5}(1 - \sqrt{2}) + \sqrt{2}(\sqrt{5} - 1)$.

108. $\sqrt{3}(1 - \sqrt{5x}) + \sqrt{5}(\sqrt{3x} - 1)$.

109. $\sqrt{7}(\sqrt{2} - 1) + \sqrt{2}(1 - \sqrt{7})$.

110. $2\sqrt{8} \cdot 3\sqrt{6}$.

111. $(2\sqrt{6} - 3\sqrt{3})(3\sqrt{6} + 4\sqrt{3})$.

112. $(-4 - 2\sqrt{7})(-4 + 2\sqrt{7})$.

113. $(\sqrt{2x} + \sqrt{50})^2$.

114. $(3\sqrt{7} - 2)(3\sqrt{7} + 2)$.

115. $(3 - 2\sqrt{3})^2 - 2(3 - 2\sqrt{3}) - 2$.

116. $(1 + \sqrt{2} - \sqrt{3})(1 - \sqrt{2} + \sqrt{3})$.

117. $(3 + 5\sqrt{2})(2\sqrt{5} - 3\sqrt{8} + \sqrt{6})$.

Criticise the errors made in the following.

118. $\dfrac{\overset{1}{\cancel{2}} + \sqrt{7}}{\underset{1}{\cancel{2}}} \overset{?}{=} 1 + \sqrt{7}$.

119. $\dfrac{\overset{1}{\cancel{a + b}} + \sqrt{a^2 - b^2}}{\underset{1}{\cancel{a + b}}} \overset{?}{=} 1 + \sqrt{a^2 - b^2}$.

120. $\dfrac{c - d \pm \sqrt{c^2 - 4cd + 4d^2}}{2c} \overset{?}{=} \dfrac{c - d \pm c - 2d}{2c}$.

Simplify the following.

121. $x = \dfrac{-(6b - a) \pm \sqrt{36b^2 + 12ab + a^2}}{6b}$.

122. $x = \dfrac{-4 \pm \sqrt{32}}{4}$.

123. $x = \dfrac{ac \pm \sqrt{a^2m + a^2p}}{4a}$.

7.9. *Rationalizing denominators*

It is possible and often desirable to move all radicals appearing in a simple fraction into the numerator. This operation is called *rationalizing the denominator.*

For comments on the illustrative examples below, see the paragraph that follows them.

Example 1. $\sqrt[3]{\dfrac{3}{2}} = \sqrt[3]{\dfrac{3 \cdot 2^2}{2 \cdot 2^2}} = \sqrt[3]{\dfrac{12}{2^3}} = \dfrac{\sqrt[3]{12}}{2}.$

Example 2.

$$\sqrt[4]{\frac{a^5}{bc^2d^7}} = \sqrt[4]{\frac{a^5(b^3c^2d)}{bc^2d^7(b^3c^2d)}} = \sqrt[4]{\frac{a^5b^3c^2d}{b^4c^4d^8}} = \frac{a\sqrt[4]{ab^3c^2d}}{bcd^2}.$$

Example 3. $\dfrac{1}{\sqrt{2a}} = \dfrac{1}{\sqrt{2a}} \cdot \dfrac{\sqrt{2a}}{\sqrt{2a}} = \dfrac{\sqrt{2a}}{2a}.$

Example 4. $\dfrac{1}{\sqrt[3]{2a}} = \dfrac{1}{\sqrt[3]{2a}} \dfrac{\sqrt[3]{(2a)^2}}{\sqrt[3]{(2a)^2}} = \dfrac{\sqrt[3]{4a^2}}{2a}.$

Example 5.

$$\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3+2\sqrt{3}+1}{(\sqrt{3})^2-(1)^2}$$

$$= \frac{4+2\sqrt{3}}{2} = \frac{\overset{1}{\cancel{2}}(2+\sqrt{3})}{\underset{1}{\cancel{2}}} = 2+\sqrt{3}.$$

Example 6. $x^{\frac{2}{3}}y^{-1}z^{-\frac{3}{2}} = \dfrac{x^{\frac{2}{3}}}{yz^{\frac{3}{2}}} = \dfrac{x^{\frac{2}{3}}z^{\frac{1}{2}}}{yz^2} = \dfrac{x^{\frac{4}{6}}z^{\frac{3}{6}}}{yz^2} = \dfrac{\sqrt[6]{x^4z^3}}{yz^2}.$

These illustrations show some of the commonly used methods. In each of Examples 1 and 2 the numerator and denominator of the radicand are multiplied by a factor which makes the denominator a perfect rth power, so that it can be taken from under the radical sign. In Examples 3 and 4, where the radicals appear first in the denominators only,

the multipliers must themselves be radicals. Example 5 makes use of the fact that $(a - b)(a + b) = a^2 - b^2$. Example 6 shows how the laws of exponents can be used to rationalize an expression which involves fractional exponents for some of its divisors in a denominator.

An important advantage of the process of rationalizing the denominator appears in numerical cases, where it shortens the work of getting decimal approximations to the values of surds.

Example. Evaluate $\dfrac{1}{\sqrt{2}}$ to four decimal places.

Long solution. $\dfrac{1}{\sqrt{2}} = \dfrac{1}{1.4142} = .7071$ (by long division).

Short solution. $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} = \dfrac{1.4142}{2} = .7071.$

While the rationalization of denominators is useful at times, the student should not get the impression that the resulting form is necessarily the "better" one. For example, $\sqrt{\dfrac{a}{b}}$ is shorter than $\dfrac{\sqrt{ab}}{b}$ and just as good for many purposes. Certainly $\dfrac{10}{\sqrt{543}}$ is preferable to $\dfrac{10\sqrt{543}}{543}$ when one is preparing to square this fraction.

EXERCISE 37

Rationalize the denominators. Approximate in decimal form the values of the numerical fractions by use of Table 2.

1. $\sqrt{\frac{1}{2}}$. **2.** $\sqrt{\frac{1}{3}}$. **3.** $\sqrt{\frac{3}{2}}$.

4. $\sqrt{\frac{3}{4}}$. **5.** $\sqrt{\frac{2}{5}}$. **6.** $\sqrt{\frac{3}{5}}$.

7. $\sqrt{\frac{4}{5}}$. **8.** $\sqrt{\frac{5}{6}}$. **9.** $\sqrt{\frac{3}{8}}$.

10. $\sqrt{\frac{5}{8}}$. **11.** $\sqrt{1 + \frac{1}{2}}$. **12.** $\sqrt{2 - \frac{1}{3}}$.

13. $\sqrt{3 - \frac{1}{5}}$. **14.** $\sqrt{4 + \frac{3}{5}}$. **15.** $\sqrt{3 - \frac{3}{5}}$.

16. $\sqrt{5 - \dfrac{2x}{3y}}$. **17.** $\sqrt{3 + \dfrac{2x^0}{5y^{-1}}}$. **18.** $\sqrt{1 - \dfrac{3x^{-2}}{5y^2}}$.

19. $\sqrt[3]{\frac{1}{2}}$.
20. $\sqrt[3]{\frac{1}{3}}$.
21. $\sqrt[3]{\frac{2}{3}}$.
22. $\sqrt[3]{\frac{3}{4}}$.
23. $\sqrt{1 + \frac{1}{x}}$.
24. $\sqrt{2 - \frac{3}{x}}$.
25. $\sqrt{x + \frac{1}{x}}$.
26. $\sqrt{x - \frac{2}{x}}$.
27. $\sqrt{x - \frac{1}{x}}$.
28. $\sqrt{3x - \frac{1}{x}}$.
29. $\sqrt{\frac{3}{x} - x}$.
30. $\sqrt{\frac{2}{x} + x}$.
31. $\sqrt{\frac{1}{x} - x^2}$.
32. $\sqrt{x - \frac{2}{x^3}}$.
33. $\sqrt{x^2 + \frac{1}{x^3}}$.

Change to radical form and rationalize the denominator.

34. $x^{\frac{1}{2}}y^0z^{-\frac{3}{2}}$.
35. $x^0y^{\frac{3}{2}}z^{-\frac{1}{2}}$.
36. $(3^{-\frac{1}{2}}x^{-\frac{3}{2}}y^0)(x^{\frac{3}{2}}y^{\frac{1}{2}})$.
37. $\dfrac{3x^{\frac{1}{2}}y^0}{z^{\frac{3}{2}}}$.
38. $\dfrac{2x^{-\frac{1}{2}}y^{\frac{3}{2}}z^0}{3x}$.
39. $\dfrac{4^0x^{-\frac{1}{2}}y}{3xy^{\frac{1}{2}}}$.
40. $\dfrac{5^{\frac{1}{2}}x^0y^{-\frac{5}{2}}}{3x^{-\frac{3}{2}}y}$.

Rationalize the denominators and simplify.

41. $\dfrac{1+\sqrt{2}}{\sqrt{2}}$.
42. $\dfrac{1-\sqrt{3}}{\sqrt{3}}$.
43. $\dfrac{2+\sqrt{5}}{\sqrt{5}}$.
44. $\dfrac{3-\sqrt{2}}{\sqrt{2}}$.
45. $\dfrac{5-\sqrt{3}}{\sqrt{3}}$.
46. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
47. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
48. $\dfrac{2}{1+\sqrt{3}}$.
49. $\dfrac{3}{1-\sqrt{2}}$.
50. $\dfrac{\sqrt{2}}{\sqrt{2}-1}$.
51. $\dfrac{\sqrt{3}}{\sqrt{3}+1}$.
52. $\dfrac{5}{\sqrt{3}-1}$.
53. $\dfrac{1}{\sqrt{3x}}$.
54. $\dfrac{3}{\sqrt{3ab}}$.
55. $\dfrac{2}{\sqrt{a+b}}$.
56. $\dfrac{2}{\sqrt{a}+\sqrt{b}}$.
57. $\dfrac{1}{\sqrt[3]{3x^2}}$.
58. $\dfrac{1}{\sqrt[4]{8x}}$.
59. $\dfrac{5}{\sqrt[3]{5x}}$.
60. $\sqrt{\dfrac{2xy^0}{z}}$.
61. $\sqrt[3]{\dfrac{x^{-3}y^4z^0}{xy^{-3}}}$.
62. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.
63. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}}$.
64. $\dfrac{\sqrt{ab^3}+\sqrt{a^3b}}{\sqrt{ab^3}-\sqrt{a^3b}}$.
65. $\dfrac{\sqrt{3x}+1}{\sqrt{3x+1}}$.
66. $\dfrac{\sqrt{8a}}{2} + \sqrt{\dfrac{a}{2}}$.
67. $\dfrac{\sqrt{a+b}}{\sqrt{a^2-b^2}}$.

68. $\dfrac{5\sqrt{6}+2}{3\sqrt{2}+6}$. 69. $\dfrac{3\sqrt{2}-\sqrt{3}}{2\sqrt{3}-\sqrt{2}}$. 70. $\dfrac{2\sqrt{5}-3\sqrt{2}}{3\sqrt{5}+2\sqrt{2}}$.

71–84. Rationalize the numerators in problems 41–7 and 61–7.

Simplify the following expressions, leaving the denominators rationalized.

85. $\left(\sqrt{\dfrac{2}{3}}\right)^3$ *Solution 1.* $\left(\sqrt{\dfrac{2}{3}}\right)^3 = \left(\sqrt{\dfrac{2}{3}}\right)^2\sqrt{\dfrac{2}{3}} = \dfrac{2}{3}\cdot\dfrac{\sqrt{6}}{3} = \dfrac{2\sqrt{6}}{9}.$

Solution 2. $\left(\sqrt{\dfrac{2}{3}}\right)^3 = \left[\left(\dfrac{2}{3}\right)^{\frac{1}{2}}\right]^3 = \left(\dfrac{2}{3}\right)^{\frac{3}{2}} = \dfrac{2^{\frac{3}{2}}}{3^{\frac{3}{2}}}$

$$= \frac{2^{\frac{3}{2}}\cdot 3^{\frac{1}{2}}}{3^2} = \frac{2\cdot 2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}{9} = \frac{2\sqrt{6}}{9}.$$

86. $(\sqrt{\frac{1}{2}})^3$. 87. $(\sqrt{\frac{1}{5}})^3$. 88. $(\sqrt{\frac{3}{2}})^3$. 89. $(\sqrt[3]{\frac{1}{2}})^4$.
90. $(\sqrt[3]{\frac{2}{3}})^4$. 91. $(\sqrt[3]{\frac{3}{2}})^4$. 92. $(\sqrt[3]{\frac{3}{5}})^4$. 93. $\sqrt{(\frac{2}{3})^3}$.
94. $\sqrt{(\frac{1}{2})^3}$. 95. $\sqrt{(\frac{1}{3})^5}$. 96. $\sqrt{(\frac{2}{5})^3}$. 97. $\sqrt{(\frac{3}{2})^3}$.
98. $\sqrt[3]{(\frac{1}{2})^4}$. 99. $\sqrt[3]{(\frac{2}{3})^4}$. 100. $\sqrt[3]{(\frac{2}{5})^4}$. 101. $\sqrt[3]{(\frac{3}{2})^4}$.

7.10. *Changing the order of radicals*

When the index of a radical is replaced by a smaller integer the order is said to be *reduced.* This is possible in the radical $\sqrt[r]{a^m}$ when m and r have a common factor larger than 1. The details may be carried through by changing to exponential form.

Example 1. $\sqrt[4]{4} = \sqrt[4]{2^2} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

Example 2. $\sqrt[10]{a^6} = a^{\frac{6}{10}} = a^{\frac{3}{5}} = \sqrt[5]{a^3}.$

It is possible always to equalize the orders of two radicals — a fact sometimes useful for comparison purposes.

Example. Which is larger, $\sqrt{2}$ or $\sqrt[3]{3}$?

Solution. $\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8}.$
$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$

Hence, $\sqrt[3]{3} > \sqrt{2}.$

When the orders of two radicals are equalized their product or quotient may be found as a single radical, as illustrated below.

Example 1. $\sqrt{2}\,\sqrt[3]{3} = 2^{\frac{3}{6}}3^{\frac{2}{6}} = (2^3 3^2)^{\frac{1}{6}} = \sqrt[6]{2^3 3^2} = \sqrt[6]{72}.$

Example 2. $\dfrac{\sqrt{5}}{\sqrt[4]{2}} = \dfrac{5^{\frac{1}{2}}}{2^{\frac{1}{4}}} = \dfrac{5^{\frac{2}{4}}}{2^{\frac{1}{4}}} = \sqrt[4]{\dfrac{5^2}{2}} = \dfrac{\sqrt[4]{5^2 2^3}}{2} = \dfrac{\sqrt[4]{200}}{2}.$

7.11. *Steps in simplifying a radical*

When the following steps are taken, a radical is said *by definition* to be in *simplest form.* The order of steps as indicated is satisfactory, though not necessary.

1. The radicand is made a simple fraction.
2. Negative and zero exponents are removed by use of the proper definitions.
3. The radicand is reduced to lowest terms.
4. The order is reduced as much as possible.
5. The denominator is rationalized.

In all cases involving radicals the student should keep in mind two fundamental principles:

1. Operations may be verified in cases of doubt by changing the radicals to exponential form.
2. When the processes are long and involved the practical procedure may be to use the decimal approximations for various surds.

EXERCISE 38

Change the orders of the following radicals as directed. 272

1. $\sqrt{3}$ to the 4th order.
2. $\sqrt{2}$ to the 6th order.
3. $\sqrt[3]{2}$ to the 6th order.
4. $\sqrt[3]{3}$ to the 6th order.
5. $\sqrt[4]{25}$ to the 2nd order.
6. $\sqrt[6]{8}$ to the 2nd order.
7. $\sqrt[5]{2}$ to the 10th order.
8. $\sqrt[10]{32}$ to the 2nd order.
9. $\sqrt[9]{27}$ to the 3rd order.
10. $\sqrt[6]{36}$ to the 2nd order.

Find the larger quantity in each pair.

11. $\sqrt{3}$, $\sqrt[3]{5}$. **12.** $\sqrt{5}$, $\sqrt[3]{11}$. **13.** $\sqrt[3]{2}$, $\sqrt[4]{3}$.

14. $\sqrt[3]{2}$, $\sqrt[6]{5}$. **15.** $\sqrt[4]{2}$, $\sqrt[6]{3}$.

Reduce the radicals in each problem below to the same order, and then perform the indicated operations.

16. $\sqrt{2}\,\sqrt[3]{2}$. **17.** $\sqrt{3}\,\sqrt[3]{3}$. **18.** $\sqrt[3]{2}\,\sqrt{3}$. **19.** $\sqrt[4]{2}\,\sqrt{2}$.

20. $\sqrt[6]{9}\,\sqrt[3]{2}$. **21.** $\sqrt{3}\,\sqrt[6]{8}$. **22.** $\dfrac{\sqrt[6]{8}}{\sqrt{2}}$. **23.** $\dfrac{\sqrt[4]{2}}{\sqrt{2}}$.

24. $\dfrac{\sqrt[3]{2}}{\sqrt[6]{9}}$. **25.** $\dfrac{\sqrt[10]{4}}{\sqrt[5]{2}}$. **26.** $\dfrac{\sqrt[6]{9}}{\sqrt[3]{3}}$. **27.** $\dfrac{\sqrt{3}}{\sqrt[6]{9}}$.

28. $\dfrac{\sqrt[3]{x}}{\sqrt{x}}$. **29.** $\dfrac{\sqrt[3]{x^2}}{\sqrt{x}}$. **30.** $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$. **31.** $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^4}}$.

Express in simplest form.

32. $\sqrt{x^2 - \dfrac{x^4}{4}}$. **33.** $\sqrt[4]{\dfrac{a^6}{9}}$. **34.** $\sqrt[3]{\dfrac{x^4y^0}{4a}}$.

35. $\sqrt[5]{1 - \dfrac{x^5}{y^5}}$. **36.** $\sqrt[3]{\dfrac{3x^0y}{2x^{-1}y^{-2}}}$. **37.** $\sqrt[4]{\dfrac{2x^0y^{-1}z^{-2}}{27y^2z}}$.

Chapter Eight

THE NUMBER SYSTEM

8.1. *Imaginary numbers*

Thus far all numbers we have met, whether rational or irrational, have been *real*, or representable by points on the line of Fig. 1. To get the point representing the irrational number $\sqrt{2}$, for instance, we simply measure to the right of the zero-point a length equal to the diagonal of a square with unit sides. But, since $2^2 = 4$ and $(-2)^2 = 4$, the quantity $\sqrt{-4}$, or the number whose square is -4, just does not exist among the real numbers. It turns out to be highly useful in mathematics to invent a new type of number in terms of which we can express such quantities as $\sqrt{-4}$.

By definition, then, the quantity $\sqrt{-1}$ is designated by the letter i. That is, $i^2 = -1$, and *i is the principal square root of -1*. It is also called the *imaginary unit.*

The other square root of -1 is $-\sqrt{-1}$, or $-i$.

Among the rules for operations with radicals (Art. 7.6) is the rule that $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. Hence, $\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\,\sqrt{a} = i\sqrt{a}$. That is,

$$\sqrt{-a} = i\sqrt{a}. \tag{1}$$

Examples.

$$\sqrt{-4} = i\sqrt{4} = 2i;$$
$$\sqrt{-7} = i\sqrt{7};$$
$$-\sqrt{-9} = -\,i\sqrt{9} = -3i.$$

Since $(-2)^4 = (+2)^4 = 16$, $\sqrt[4]{-16}$ cannot be either $+2$ or -2. In general, it can be shown that all *even roots of negative numbers* are imaginary, or expressible in terms of

the imaginary unit i. There are still other quantities which can be so expressed, but we shall not consider them here.

Definition. An imaginary number is one which can be written in the form $a + bi$, where a and b are real and $b \neq 0$.

Examples. i, $-\frac{3i}{2}$, $4 + 5i$, $6 - i$, $\sqrt{-4}$, $\sqrt[6]{-2}$.

Imaginary numbers should be expressed in terms of i before they are used in algebraic operations. For instance, it is false that $\sqrt{-3}\sqrt{-3} = \sqrt{(-3)(-3)} = \sqrt{9} = 3$, since from the definition of the square root, $(\sqrt{-3})^2 = -3$. But if we write $\sqrt{-3}$ as $i\sqrt{3}$, then

$$(\sqrt{-3})^2 = i^2(\sqrt{3})^2 = (-1)(3) = -3.$$

8.2. *Powers of i*

Any power of i is equal to one of the four numbers i, -1, $-i$, and 1. This conclusion is reached as follows:

$i^1 = i.$

$i^2 = -1$ (by definition).

$i^3 = i^2 i = (-1)i = -i.$

$i^4 = i^3 i = (-i)i = -i^2 = 1.$

$i^5 = i^4 i = 1 \cdot i = i.$

$i^6 = i^4 i^2 = 1(-1) = -1.$

$i^7 = i^4 i^3 = 1(-i) = -i.$

$i^8 = i^4 i^4 = 1 \cdot 1 = 1.$

$i^9 = i^8 i = 1 \cdot i = i$, etc.

Note that $i^5 = i^1$, $i^6 = i^2$, $i^7 = i^3$, $i^8 = i^4$, $i^9 = i^1$, etc.

Rule for finding the value of any power of i. Divide the exponent by 4. The remainder will be 0, 1, 2, or 3. The corresponding values of the power are $i^0 = 1$, $i^1 = i$, $i^2 = -1$, and $i^3 = -i$.

Examples. $i^{20} = 1$, $i^{25} = i^1 = i$, $i^{30} = i^2 = -1$, $i^{27} = i^3 = -i$.

8.3. *Operations with imaginary numbers*

The results of simple operations with imaginary numbers can be expressed in the form $a + bi$, as in the illustrative examples below.

Addition.

$$\begin{aligned}(2+3i)+(4-5i) &= (2+4)+(3i-5i) = 6-2i\\ &= 6+(-2)i.\end{aligned}$$

Subtraction.

$$\begin{aligned}(3-7i)-(4-9i) &= (3-7i)+(-4+9i)\\ &= (3-4)+(-7i+9i) = -1+2i.\end{aligned}$$

Multiplication.

$$\begin{aligned}(3+2i)(4-i) &= 12+8i-3i-2i^2\\ &= 12+5i-2(-1) = 14+5i.\end{aligned}$$

Division.

$$\begin{aligned}\frac{2-i}{1+3i} &= \frac{(2-i)(1-3i)}{(1+3i)(1-3i)} = \frac{-1-7i}{1-9i^2}\\ &= \frac{-1-7i}{1+9} = -\frac{1}{10}+\left(-\frac{7}{10}\right)i.\end{aligned}$$

Note that $(a+bi)(a-bi) = a^2 - b^2i^2 = a^2 + b^2$ rather than $a^2 - b^2$.

EXERCISE 39

Reduce each of the following expressions to simplest form.

1. $\sqrt{-25}$. *Answer:* $5i$.
2. $\sqrt{-17}$. *Answer:* $i\sqrt{17}$.
3. $\sqrt{-4x^2}$. *Answer:* $2ix$.
4. $\sqrt{-36}$.
5. $\sqrt{-7}$.
6. $\sqrt{-9}$.
7. $\sqrt{-49}$.
8. $\sqrt{-47}$.
9. $\sqrt{-64}$.
10. $\sqrt{-19}$.
11. $\sqrt{-144}$.
12. $-\sqrt{-4}$.
13. $-\sqrt{-9}$.
14. $-\sqrt{-11}$.
15. $-\sqrt{-25}$.
16. $\sqrt{-9a^2}$.
17. $\sqrt{-16x^2}$.
18. $-\sqrt{-25y^4}$.
19. $\sqrt{-17x^4}$.
20. $-\sqrt{-5x^2}$.
21. $-\sqrt{-23a^2}$.
22. i^{10}.
23. i^{11}.
24. i^{14}.
25. i^{26}.
26. i^{29}.
27. $(1+i)^2$.
28. $(1-i)^2$.
29. $(2+i)^2$.
30. $(2-i)^2$.
31. $(1-2i)^2$.
32. $(3+2i)(3-2i)$.
33. $(5+4i)(5-4i)$.
34. $(2+3i)(3-i)$.
35. $(4+i)(4-3i)$.
36. $(6+5i)(6-5i)$.
37. $i(3+4i)$.
38. $3i(3-i)$.
39. $2(3-i)^2$.
40. $\dfrac{3}{2+i}$.

41. $\dfrac{2}{3-i}$. 42. $\dfrac{3}{4+i}$. 43. $\dfrac{2}{3+2i}$.

44. $\dfrac{5}{3-2i}$. 45. $\dfrac{1+i}{1-i}$. 46. $\dfrac{1-i}{1+i}$.

47. $\dfrac{2+3i}{3-i}$. 48. $\dfrac{2-3i}{3+i}$. 49. $\dfrac{3+2i}{2+i}$.

50. $\dfrac{3-4i}{4+3i}$. 51. $\dfrac{4+3i}{3-4i}$. 52. $\dfrac{3}{2i}$.

53. $\dfrac{-2}{3i}$. 54. $\dfrac{-1}{i}$. 55. $\dfrac{3}{5i^7}$.

56. $\dfrac{-7}{3i^5}$. 57. $\dfrac{10}{3i^3}$. 58. $\dfrac{6}{2i^9}$.

59. $\dfrac{5}{3i^{11}}$. 60. $\sqrt{-5}\sqrt{-5}$. 61. $\sqrt{-3}\sqrt{-12}$.

62. $\sqrt{-4}(5+\sqrt{-25})$. 63. $\sqrt{-4}+\sqrt{-16}$.

64. $\sqrt{-3}+2\sqrt{-12}$. 65. $\sqrt{-2}\sqrt{-3}\sqrt{-5}$.

66. $\dfrac{\sqrt{-10}}{\sqrt{-5}}$. 67. $\dfrac{\sqrt{-10}}{\sqrt{5}}$.

68. $\dfrac{5}{2+\sqrt{-9}}$. 69. $\dfrac{\sqrt{4}+\sqrt{-4}}{3\sqrt{-16}}$.

70. $\dfrac{2\sqrt{-9}}{\sqrt{9}+\sqrt{-9}}$. 71. $\dfrac{\sqrt{4}-\sqrt{-4}}{\sqrt{-4}}$.

8.4 *

We are now prepared to examine again this thing called a *number*. To represent all of its types thus far met we shall need to extend Fig. 1 as shown in Fig. 13. The line of Fig. 1, now called the *axis of reals*, contains all numbers except imaginary ones. Represented by points on this line are *integers* (0, 1, 2, −1, −2, etc.), *fractions* ($\frac{1}{2}$, $\frac{8}{5}$, $-\frac{7}{9}$, $-\frac{16}{3}$, etc.), *decimal numbers* (0.5, .33, −1.21, etc.), and *irrational numbers* ($\sqrt{2}$, $\sqrt[4]{7}$, $-2+\sqrt[3]{16}$, π, etc.). But there is no place on this line for imaginary numbers such as i and $1-2i$. The place is provided in a very simple and neat

* This article may be omitted without loss of continuity. If it is studied, Article 7.4 should be re-read first.

manner by means of the plane (that of the paper) containing the axis of reals. In Fig. 13 the vertical line is called the *axis of imaginaries*. It contains all numbers, such as i, $-2i$, and $\sqrt{7}i$, which have the form $0 + bi$, where b is real and not

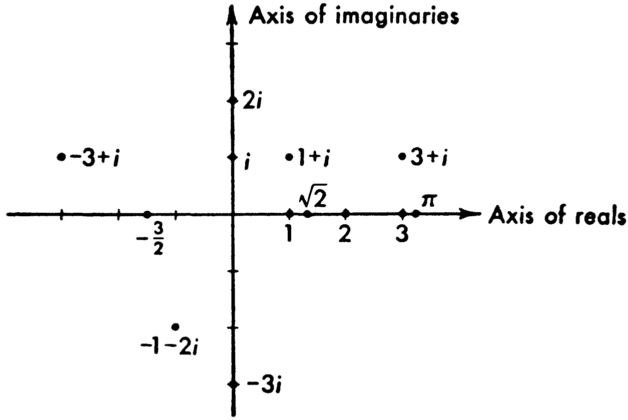

Fig. 13

zero. Such numbers are called *pure imaginaries*. The general number $a + bi$, where a and b are real, is represented by a point whose coordinates would be (a, b) if the axes of reals and imaginaries were respectively the X and Y axes of the rectangular coordinate plane. Thus the point representing the number $1 - 2i$ has the rectangular coordinates $(1, -2)$.

The numbers represented in their totality by all of the points on the plane are called *complex numbers*. The plane containing them is known as the *complex plane*.

8.5. *Classifications of numbers*

The following diagram is helpful.

Numbers in General, or Complex Numbers

Real Numbers | Imaginary Numbers

Real Numbers:

Classification 1
- Zero
- Positive numbers
- Negative numbers

Classification 2
- Rational numbers
- Irrational numbers

In this diagram Classification 1 divides the real numbers, represented by points on the axis of reals, into three groups corresponding with: (1), the point 0 (zero); (2), points to the right of 0; and (3), points to the left of 0.

Classification 2 is simply another grouping of real numbers. Here algebra and reason have stepped beyond geometry, since the points representing rational and irrational numbers are mixed together too thickly in any segment of the axis of reals to be separated visually, though we may find sample points representing each of the two types.

8.6. *Numbers as roots of equations*

It is interesting that the requirements for roots of rational integral equations in one unknown, which appear in the solutions of very simple problems, have called into use all the types of numbers thus far discussed. The following table illustrates the point.

	Equation	Root or roots	Type of number represented by the root
(1)	$x = 0.$	0	A type by itself
(2)	$x - 3 = 0.$	3	Positive integer
(3)	$x + 2 = 0.$	-2	Negative integer
(4)	$2x - 1 = 0.$	$\frac{1}{2}$	Positive fraction
(5)	$3x + 5 = 0.$	$-\frac{5}{3}$	Negative fraction
(6) *	$x^2 - 2 = 0.$	$\pm\sqrt{2}$	Irrational numbers
(7) *	$x^2 + 2 = 0.$	$\pm i\sqrt{2}$	Pure imaginary numbers
(8) *	$x^2 + x + 2 = 0.$	$\frac{-1 \pm i\sqrt{7}}{2}$	Imaginary numbers

EXERCISE 40

1. Show on a complex plane the points representing the following numbers: (a), 2; (b), $2 + i$; (c), $3i$; (d), $1 - i$; (e), $-i$; (f), $-2 + 3i$; (g), $4 - i$; (h), $\frac{-5}{2}$; (i), $\frac{i}{2}$; (j), $\frac{3}{2} + \frac{i\sqrt{3}}{2}$; (k), 0; (l), $-3 - 2i$.

* The solutions of these equations will be discussed in Chapter 9.

2. The numbers $a + bi$ and $a - bi$ are called *conjugate imaginaries*. What is the relation between the points which represent them on the complex plane?

3. In the light of the discussion in this chapter, how would you answer the following questions?

(a) How many real numbers are there between 0 and 1?
(b) Is there a smallest positive number, and, if so, what is it?
(c) Is there a largest positive number which is less than 1? If so, what is it?

Chapter Nine

QUADRATIC EQUATIONS

9.1. *The quadratic in standard form*

A *quadratic equation*, or simply a *quadratic*, in a given letter is a rational integral equation of the second degree in that letter.

Example 1. $2x^2 - 3x + 1 = 0$ (quadratic in x).

Example 2. $3y^2 - 2ay + by^2 + c = 0$ (quadratic in y).

When terms of the same degree are grouped together, a quadratic in x may be written in the *standard form*

$$(1) \qquad ax^2 + bx + c = 0, \qquad a \neq 0,$$

where the coefficients a, b, and c do not involve x.

Example. Write $5x^2 - 7x + 3 = 2x^2 + 3x - 4$ in standard form.

Solution. Grouping, the terms, we have

$$5x^2 - 2x^2 - 7x - 3x + 3 + 4 = 0,$$

or

$$(5 - 2)x^2 + (-7 - 3)x + (3 + 4) = 0,$$

or

$$(2) \qquad 3x^2 - 10x + 7 = 0.$$

Comparing (2) with standard form (1), we find that $a = 3$, $b = -10$, and $c = 7$.

9.2. *The roots of a quadratic equation*

We have seen that a root of an equation is a number or expression which satisfies the equation when substituted for the unknown. Thus, for the quadratic

$$(1) \qquad x^2 + 4x - 5 = 0,$$

one root is -5, since $(-5)^2 + 4(-5) - 5 = 0$; and a second root is 1, since $1^2 + 4 \cdot 1 - 5 = 0$.

The roots of

$$(2) \qquad x^2 + 4x - 1 = 0$$

are $-2 + \sqrt{5}$ and $-2 - \sqrt{5}$. Checking the first one, we find that $(-2 + \sqrt{5})^2 + 4(-2 + \sqrt{5}) - 1 = 4 - 4\sqrt{5} + 5 - 8 + 4\sqrt{5} - 1 = 0$. The test of the root $-2 - \sqrt{5}$ is left to the student.

Again, the quadratic

$$(3) \qquad x^2 + 4x + 4 = 0$$

is satisfied by $x = -2$; but in this case there is no second root different from the first one. For reasons to be discussed later we shall say that -2 is a *double* root of (3).

Finally,

$$(4) \qquad x^2 + 4x + 8 = 0$$

is satisfied by the imaginary roots $-2 + 2i$ and $-2 - 2i$, where i is the imaginary unit (Art. 8.1). For $(-2 + 2i)^2 + 4(-2 + 2i) + 8 = 4 - 8i + 4i^2 - 8 + 8i + 8 = 4 + 4i^2 = 4 + 4(-1) = 0$. Similarly, $-2 - 2i$ satisfies (4).

The roots of equations (1)–(4) represent among them the four different kinds of root-pairs for quadratics — rational and unequal, irrational, rational and equal, and imaginary. In Art. 9.9 we shall discuss a geometric interpretation of the various types. As may be suspected here, and will be proved later, *every quadratic equation has two roots,* though in some cases they are equal.

EXERCISE 41

Write each of the following equations in standard form. Comparing it with: $ax^2 + bx + c = 0$, find the value of a, b, and c in each case. Prove that each of the numbers listed is a root.

1. $3x^2 - 2x = 1$; $(1, -\frac{1}{3})$. *Solution.* $3x^2 - 2x - 1 = 0$, or $3x^2+(-2)x+(-1)=0$; $a=3$, $b=-2$, and $c=-1$; $3 \cdot 1^2-2 \cdot 1=1$; $3(-\frac{1}{3})^2 - 2(-\frac{1}{3}) = 1$.

2. $4x = 1 + 4x^2$; $(\frac{1}{2})$.
3. $6 = 2x^2 + x$; $(-2, \frac{3}{2})$.
4. $1 = 3x^2 + 2x$; $(-1, \frac{1}{3})$.
5. $-3x = 2x^2 + 1$; $(-1, -\frac{1}{2})$.
6. $3 = x^2 + 2x$; $(1, -3)$.
7. $6 = 5x + 4x^2$; $(-2, \frac{3}{4})$.
8. $3 = x + 2x^2$; $(1, -\frac{3}{2})$.
9. $3x = -2 - x^2$; $(-1, -2)$.
10. $5x = 2x^2 + 2$; $(2, \frac{1}{2})$.
11. $4x = 3 - 4x^2$; $(\frac{1}{2}, -\frac{3}{2})$.
12. $8x = -3 - 4x^2$; $(-\frac{1}{2}, -\frac{3}{2})$.
13. $2x = x^2$; $(0, 2)$.
14. $x^2 = 3x$; $(0, 3)$.
15. $2x^2 = 3x$; $(0, \frac{3}{2})$.
16. $2x = -3x^2$; $(0, -\frac{2}{3})$.
17. $3x = -2x^2$; $(0, -\frac{3}{2})$.
18. $d^2 = x^2$; $(d, -d)$.
19. $4x^2 = 9e^4$; $\left(\frac{3e^2}{2}, -\frac{3e^2}{2}\right)$.
20. $x^2-(r+s)x+rs=0$; (r, s).
21. $4d^2x^2 = e^2$; $\left(\frac{e}{2d}, \frac{-e}{2d}\right)$.
22. $x^2+(r-s)x-rs=0$; $(-r, s)$.
23. $a^2x^2 = 16b^2$; $\left(\frac{4b}{a}, \frac{-4b}{a}\right)$.

Write each of the following equations in standard form and find the expressions corresponding with a, b, and c.

24. $Ax^2 + x - Bx = C + x^2 - 4$. *Answer.* $(A - 1)x^2 + (1 - B)x + (4 - C) = 0$; $a = A - 1$; $b = 1 - B$; $c = 4 - C$.

25. $3x^2+4x+2=Ax^2+Bx+C$.
26. $2x^2-Ax+4=Cx^2+Bx-A$.
27. $3x^2+Bx-x=Ax^2-Cx+1$.
28. $Ax^2+B+Cx=Bx^2-Ax+C$.
29. $Ax^2 + B = Cx^2 + Dx$.
30. $ex^2 + 2Bx = 4 - x + Ax$.

9.3. *The solution of* $ax^2 + c = 0$

When $b = 0$, the general quadratic equation (1), Art. 9.1, takes the form

(1) $$ax^2 + c = 0.$$

Since (1) is linear in the unknown x^2, it follows that

$$(2) \qquad x^2 = -\frac{c}{a},$$

and hence,

$$(3) \qquad x = \sqrt{-\frac{c}{a}} \quad \text{or} \quad x = -\sqrt{-\frac{c}{a}}.$$

Example 1. If $4x^2 - 9 = 0$, $x^2 = \frac{9}{4}$, and $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

Example 2. If $2x^2 - 3 = 0$, $x^2 = \frac{3}{2}$, and $x = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$ or $x = -\sqrt{\frac{3}{2}} = -\frac{\sqrt{6}}{2}$.

Example 3. If $4x^2 + 5 = 0$, $x^2 = -\frac{5}{4}$, and $x = \frac{i\sqrt{5}}{2}$ or $x = \frac{-i\sqrt{5}}{2}$.

Example 4. Solve: $\frac{3}{5x^2 - 1} = \frac{2}{1 + 4x^2}$.

Solution. Multiply both members of the equation by $(5x^2 - 1)(1 + 4x^2)$. The new equation becomes $3 + 12x^2 = 10x^2 - 2$, or $2x^2 = -5$. Hence $x^2 = -\frac{5}{2}$, and $x = i\frac{\sqrt{10}}{2}$ or $x = -i\frac{\sqrt{10}}{2}$.

9.4. *Solving quadratics by factoring*

When the left member of (1), Art. 9.1, is factorable, and when the factors can be found readily, the equation can usually be solved most easily by the method illustrated below.

Example 1. Solve: $x^2 + 4x - 5 = 0$.

Solution. First we factor the left member, thus:

$$(1) \qquad (x - 1)(x + 5) = 0.$$

This equation is satisfied by $x = 1$, since $(1 - 1)(1 + 5) = 0 \cdot 6 = 0$; and also by $x = -5$, since $(-5 - 1)(-5 + 5) = (-6)(0) = 0$. Hence 1 and -5 are roots of (1). We get $x = 1$ by setting the factor $x - 1$ equal to zero. Similarly, $x + 5 = 0$ gives $x = -5$.

Example 2. Solve: $4x^2 + 12x + 9 = 0$.

Solution. Noting that the left member is a perfect square, we have

(2) $$(2x + 3)^2 = 0, \quad \text{or} \quad (2x + 3)(2x + 3) = 0.$$

Each factor, placed equal to zero, yields $x = -\frac{3}{2}$. This explains the algebraic basis for calling $-\frac{3}{2}$ a *repeated* or *double* root. The geometric reason will appear later.

CAUTION. The solution of a quadratic by factoring the left member is correct in principle *only* when the right member is zero. For example, as a test will show, the roots of the quadratic

(3) $$(x - 2)(x - 3) = 12$$

are *not* found by setting $x - 2$ or $x - 3$ equal to 12. The explanation is left to the student. The correct roots of (3), which are 6 and -1, are found when 12 is transposed and the *new* left member is simplified and factored.

Example 3. Solve: $\dfrac{7}{4} - \dfrac{6}{2x - 1} = \dfrac{2x - 3}{2x + 3}$.

Solution. After clearing fractions by multiplying through by the L.C.D. $4(2x - 1)(2x + 3)$, we get

(4) $$7(2x - 1)(2x + 3) - 24(2x + 3) = 4(2x - 3)(2x - 1),$$

which simplifies, after the indicated operations are performed, to

(5) $$4x^2 + 4x - 35 = 0.$$

The factors of the left member of (5) are $2x + 7$ and $2x - 5$, so that $x = -\frac{7}{2}$ or $\frac{5}{2}$. Each of these values for x will be found to satisfy the original equation.

EXERCISE 42

Solve by the method of Art. 9.3 where possible; otherwise by the factoring method.

1. $2x^2 - 8 = 0.$
2. $3x^2 - 12 = 0.$
3. $5x^2 - 20 = 0.$
4. $2x^2 - 18 = 0.$
5. $2x^2 - 98 = 0.$
6. $4x^2 - 100 = 0.$
7. $3x^2 + 27 = 0.$
8. $2x^2 + 8 = 0.$
9. $3x^2 + 12 = 0.$
10. $2x^2 + 18 = 0.$
11. $2x^2 + 98 = 0.$
12. $4x^2 + 100 = 0.$
13. $2x^2 - 36 = 0.$
14. $3x^2 - 36 = 0.$
15. $2x^2 - 100 = 0.$
16. $2a^2x^2 - 64 = 0.$
17. $2b^2x^2 - 24 = 0.$
18. $a^2x^2 - 96c^2 = 0.$
19. $2x^2 + 36a^2 = 0.$
20. $3x^2 + 36b^2 = 0.$
21. $a^2x^2 + 100b^2 = 0.$
22. $2x^2 + 64b^2 = 0.$
23. $2x^2 + 48c^2 = 0.$
24. $a^2x^2 + 96b^2 = 0.$
25. $ax^2 + 25b = 0.$
26. $ax^2 + 100c = 0.$
27. $20b = -ax^2.$
28. $(x - 2)(x + 3) = 6.$
29. $(x + 2)(x - 3) = 6.$
30. $x(x + 5) = 14.$
31. $(x - 1)(x - 2) = 6.$
32. $(x + 1)(x - 2) = 4.$
33. $x(x - 3) = 10.$
34. $2x^2 + x - 1 = 0.$
35. $3x^2 + 2x - 1 = 0.$
36. $3x^2 - 2x - 1 = 0.$
37. $4x^2 - 3x - 1 = 0.$
38. $4x^2 - 4x + 1 = 0.$
39. $9x^2 + 6x + 1 = 0.$
40. $4x^2 + 12bx + 9b^2 = 0.$
41. $x^2 - 8ax + 16a^2 = 0.$
42. $9x^2 + 30x + 25 = 0.$
43. $6x^2 - x = 0.$
44. $9x^2 - 3x = 0.$
45. $8x^2 + 10x = 0.$
46. $10x^2 + ax = 0.$
47. $ax^2 + bx = 0.$
48. $cx^2 - dx = 0.$
49. $6x^2 + x = 2.$
50. $8x^2 + 10x = 3.$
51. $10x^2 - 13x = 3.$
52. $10x^2 - x = 2.$
53. $15x^2 + x = 2.$
54. $x = 2 - 15x^2.$
55. $11x = 6 + 3x^2.$
56. $9x = -2 - 10x^2.$
57. $9x = 2 + 10x^2.$
58. $23x = -6 - 21x^2.$
59. $23x = 6 + 21x^2.$
60. $25x = -6 - 14x^2.$
61. $\dfrac{2x}{5 - x} = \dfrac{15}{2} - \dfrac{2x - 10}{x}.$
62. $\dfrac{5}{2x^2 - 3} = \dfrac{7}{1 - 2x^2}.$
63. $\dfrac{15}{x + 5} + \dfrac{12}{x + 10} = 1.$
64. $(2x-3)(2x+3)=7x^2-4.$
65. $\dfrac{x + 5}{x - 2} + \dfrac{5x - 2}{3x - 8} = 0.$

66. $\dfrac{1}{4x^2 + 5} + \dfrac{1}{4x^2 - 5} = \dfrac{3}{16x^4 - 25}.$

67. $\dfrac{x + 1}{x - 2} + \dfrac{2}{5(2 - x)} = -\dfrac{1 - x}{x + 2}.$

68. $10x^2 = 7x.$

69. $\dfrac{5}{x} = \dfrac{11}{x^2}.$

70. $8x^2 - 3x = 0.$

9.5. *Solving quadratics by completing the squares*

The algebraic methods of solving quadratics thus far discussed, while short and efficient, cannot always be applied. The method called "completing the square" has the advantage that it will solve all types of quadratic equations, besides giving practice in an operation which is useful elsewhere in mathematics.

To illustrate, consider the equation

$$3x^2 - 2x - 2 = 0. \tag{1}$$

Step 1. Divide both members by 3:

$$x^2 - \frac{2x}{3} - \frac{2}{3} = 0. \tag{2}$$

Step 2. Transpose the constant term:

$$x^2 - \frac{2x}{3} = \frac{2}{3}. \tag{3}$$

Step 3. Add to both sides the number which will make the left member a perfect square. Note that in the perfect square $(x-a)^2 = x^2 - 2ax + a^2$, the third term, or a^2, is $[(\frac{1}{2})(-2a)]^2$. Hence we add to both members of the equation the square of *one-half the coefficient of* x, or $[(\frac{1}{2})(-\frac{2}{3})]^2$ in this case, yielding

$$x^2 - \frac{2x}{3} + \left(-\frac{1}{3}\right)^2 = \frac{2}{3} + \frac{1}{9}, \tag{4}$$

or

$$\left(x - \frac{1}{3}\right)^2 = \frac{7}{9}. \tag{5}$$

Note that the number added to complete the square is *always positive*, and that the middle sign of the left member of (5) is the same as the sign before the term of first degree in (4).

Step 4. Solve for $x - \frac{1}{3}$:

$$x - \frac{1}{3} = \pm\sqrt{\frac{7}{9}} = \pm\frac{\sqrt{7}}{3}. \tag{6}$$

Step 5. Transpose the term $-\frac{1}{3}$:

$$x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}, \quad \text{or} \quad x = \frac{1 \pm \sqrt{7}}{3}. \tag{7}$$

Equation (7) is a brief way of stating that

$$x = \frac{1 + \sqrt{7}}{3}, \quad \text{or} \quad x = \frac{1 - \sqrt{7}}{3}. \tag{8}$$

Note that the various steps yield equivalent equations, and that the two equations in (6), (7), and (8) are equivalent to the one equation (5). In other words, each of the two values of x shown in (8) satisfies (1), as the student may verify.

If, at the stage in the solution represented by equation (5), the right side is negative, the roots will be imaginary. For example, given

$$x^2 + 2x + 4 = 0, \tag{9}$$

then

$$x^2 + 2x + 1 = -4 + 1; \tag{10}$$

$$(x + 1)^2 = -3; \tag{11}$$

$$x + 1 = \pm\sqrt{-3} = \pm i\sqrt{3}; \tag{12}$$

$$x = -1 \pm i\sqrt{3}. \tag{13}$$

9.6. *The sum and product of the roots*

To get a result which will prove useful for checking purposes, we note that

(1) $$ax^2 + bx + c = 0$$

is equivalent to

(2) $$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

obtained from (1) by dividing the coefficients by a. Now

(3) $$x^2 - (r + s)x + rs = 0$$

is satisfied by $x = r$ and by $x = s$, as may be verified by trial. Hence if the general equation (1) is also satisfied by the roots r and s, it follows, by comparison of (3) with (2), that

(4) $$r + s = -\frac{b}{a},$$

and

(5) $$rs = \frac{c}{a}.$$

In words, *the sum of the roots of* (1) *is* $-\frac{b}{a}$, *and their product is* $\frac{c}{a}$.

This result may be used in checking the roots found in the solution of any quadratic. The check is often shorter than that obtained by substitution of the roots in the original equation, particularly when the roots are irrational or imaginary.

Example 1. By use of the sum and product formulas, show that $\frac{1 + \sqrt{11}}{2}$ and $\frac{1 - \sqrt{11}}{2}$ are roots of the equation,

$$2x^2 - 2x - 5 = 0.$$

Solution.

$$-\frac{b}{a} = -\frac{(-2)}{2} = 1,$$

and

$$\frac{1+\sqrt{11}}{2}+\frac{1-\sqrt{11}}{2}=1.$$

Also,
$$\frac{c}{a}=-\frac{5}{2},$$

and $$\left(\frac{1+\sqrt{11}}{2}\right)\left(\frac{1-\sqrt{11}}{2}\right)=\frac{1^2-\sqrt{11}^2}{4}=-\frac{10}{4}=-\frac{5}{2}.$$

Example 2. Show that $r=\dfrac{-3+i\sqrt{31}}{4}$ and $s=\dfrac{-3-i\sqrt{31}}{4}$ are roots of the equation, $2x^2+3x+5=0$.

Solution.

$$r+s=-\frac{6}{4}=-\frac{3}{2}=-\frac{b}{a}.$$

Also,

$$rs=\left(-\frac{3}{4}\right)^2-\left(\frac{i\sqrt{31}}{4}\right)^2=\frac{9}{16}-\frac{-31}{16}=\frac{40}{16}=\frac{5}{2}=\frac{c}{a}.$$

EXERCISE 43

Find the roots of each of the following equations by the method of completing the square, and check by use of the sum and product formulas.

1. $x^2-6x+8=0$.

2. $x^2+5x+6=0$.

3. $x^2+2x-8=0$.

4. $x^2-3x-10=0$.

5. $x^2-2x-8=0$.

6. $x^2+3x-10=0$.

7. $6x^2-7x-3=0$.

8. $6x^2+x-2=0$.

9. $6x^2+7x-3=0$.

10. $6x^2-x-2=0$.

11. $12x^2+7x-12=0$.

12. $8x^2-6x-9=0$.

13. $3x^2-2x-2=0$.

14. $5x^2+3x-3=0$.

15. $5x^2-5x-7=0$.

16. $3x^2+4x=5$.

17. $2x^2+6x=5$.

18. $8x^2+4x=3$.

19. $7x^2=14x-3$.

20. $9x^2=6x-5$.

21. $5x^2=-3x-4$.

22. $3x^2=-5x-4$.

23. $4x^2=-6x-3$.

24. $10x^2=-3x-2$.

25. $3x = x^2 + 7$.

26. $2x = 5 + x^2$.

27. $3x = x^2 + 6$.

28. $4x = x^2 + 6$.

29. $6x = x^2 + 2$.

30. $8x = x^2 + 3$.

31. $(x + 1)(x - 2) = 3$.

32. $(x - 2)(x + 3) = -7$.

33. $(x + 3)(x - 4) = 1$.

34–43. Check by direct substitution the answers found in problems 1–10.

Solve, and check the answers by direct substitution.

44. $\dfrac{x + 3}{x - 1} - \dfrac{3 - 2x}{2x + 2} = \dfrac{7}{2}$.

45. $2(y - 1) - \dfrac{5(y + 1) - 5}{y + 2} = 3(1 - y) - \dfrac{5 + 2(y + 3)}{3}$.

9.7. *The quadratic formula*

When we solve the general quadratic

$$ax^2 + bx + c = 0 \tag{1}$$

by completing the square, the various equivalent equations obtained in the successive steps are as follows:

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0; \tag{2}$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}; \tag{3}$$

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2; \tag{4}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}; \tag{5}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}; \tag{6}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

Equation (7) is called *the quadratic formula.* In a sense, the infinitely many quadratic equations are here solved collectively once for all. They are seen to have *two roots apiece,*

though these roots are equal when the quantity under the radical sign is zero. To get the roots of a specific quadratic it is necessary only to identify the coefficients a, b, and c, and then to substitute their values in (7). If the equation is quadratic in some letter or quantity other than x, this letter or quantity should replace x in (7).

Example. Solve the quadratic

$$3x^2 - 2x - 4 = 0. \tag{8}$$

Solution. Here $a = 3$, $b = -2$, and $c = -4$. Substituting these values in (7), we have

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-4)}}{2 \cdot 3} = \frac{2 \pm \sqrt{4 - (-48)}}{6}$$
$$= \frac{2 \pm \sqrt{52}}{6} = \frac{2 \pm 2\sqrt{13}}{6} = \frac{2(1 \pm \sqrt{13})}{6} = \frac{1 \pm \sqrt{13}}{3}.$$

That is, one root is $\dfrac{1 + \sqrt{13}}{3}$ and the other is $\dfrac{1 - \sqrt{13}}{3}$.

After getting $x = \dfrac{2 \pm 2\sqrt{13}}{6}$, the careless student is likely to write $x = \dfrac{4\sqrt{13}}{6}$ or $x = \dfrac{0\sqrt{13}}{6}$. Why are these results incorrect?

Note that if (8) were solved by completing the square, the correct simplified root would be obtained at once. This is one reason why some who work in the field of applied mathematics prefer the square-completing method to the formula. The latter method, on the other hand, is shorter for some problems, especially where literal coefficients are involved; and it also gives other useful information about quadratics, as we shall see.

9.8. *The discriminant of a quadratic equation*

The roots of the quadratic

$$ax^2 + bx + c = 0, \tag{1}$$

as given in the quadratic formula, may be written separately thus:

$$(2) \qquad r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The radicand $b^2 - 4ac$ in (2), which we may designate by D, is called the *discriminant* of the quadratic equation (1).

For example, the discriminant of

$$(3) \qquad 2x^2 - 3x - 4 = 0,$$

for which $a = 2$, $b = -3$, and $c = -4$, is $(-3)^2 - 4(2)(-4) = 9 + 32 = 41$. Hence, the roots of (3) will involve the radical $\sqrt{41}$, and will be irrational.

Having the formulas (2) in mind, we can draw certain conclusions about the roots of (1) from the value of D, assuming that a, b, and c are rational numbers.

Premise about D	Then the roots of (1) are
(a) $D > 0$ and a perfect square	real, unequal, and rational.
(b) $D > 0$ and not a perfect square	real, unequal, and irrational.
(c) $D = 0$	real, equal, and rational.
(d) $D < 0$	imaginary.

These results have immediate and practical application as an aid to the solution of quadratic equations. When, for a particular equation, D is negative or not a perfect square, it is useless to waste time on the factoring method of solution, since the roots are imaginary or irrational.

Question. Can $3x^2 - 4x - 5$ be factored?

Answer. Since $D = (-4)^2 - 4 \cdot 3(-5) = 16 + 60 = 76$ (not a perfect square), the answer, if we bar factors with irrational coefficients,* is "No." Hence, the quadratic

* Note that $3x^2 - 4x - 5 = 3\left(x - \frac{2}{3} - \frac{\sqrt{19}}{3}\right)\left(x - \frac{2}{3} + \frac{\sqrt{19}}{3}\right)$, as we discover *after* solving the equation. This type of factoring is of course not helpful in obtaining the solution.

$3x^2 - 4x - 5 = 0$ must be solved by completing the square or by use of the quadratic formula. If the latter method is used, the test-value $D = 76$ already found can be used in writing down at once the solution: $x = \dfrac{4 \pm \sqrt{76}}{6} = \dfrac{2 \pm \sqrt{19}}{3}$.

EXERCISE 44

In each of problems 1–45, determine the nature of the roots by use of the discriminant. Then solve the equation by factoring if D is a perfect square; otherwise by use of the quadratic formula. (Note that zero is a perfect square.)

1. $2x^2 - 3x + 5 = 0$.
2. $3x^2 + 5x - 2 = 0$.
3. $4x^2 - 4x + 1 = 0$.
4. $5x^2 + 2x - 3 = 0$.
5. $x^2 - 3x + 2 = 0$.
6. $x^2 - 6x - 9 = 0$.
7. $x^2 + x - 3 = 0$.
8. $x^2 + 3x - 1 = 0$.
9. $x^2 - 3x + 1 = 0$.
10. $x^2 - 2x + 2 = 0$.
11. $x^2 - x + 3 = 0$.
12. $9x^2 - 30x + 25 = 0$.
13. $x^2 + x + 1 = 0$.
14. $2x^2 - 5x + 1 = 0$.
15. $3x^2 + x - 3 = 0$.
16. $4x^2 + x - 1 = 0$.
17. $x^2 - 10x + 1 = 0$.
18. $10y^2 - y + 1 = 0$.
19. $3x^2 - 5x + 1 = 0$.
20. $3x^2 - 7x + 2 = 0$.
21. $4y^2 - 5y + 1 = 0$.
22. $7x^2 - 8x + 1 = 0$.
23. $9x^2 - 3x + 2 = 0$.
24. $6y^2 - 5y + 1 = 0$.
25. $5x^2 - 6x + 1 = 0$.
26. $8x^2 - 7x - 1 = 0$.
27. $3y^2 - 9y + 2 = 0$.
28. $5x^2 - 4x - 1 = 0$.
29. $10x^2 - 3x - 1 = 0$.
30. $3y^2 - 9y + 2 = 0$.
31. $7x^2 - 3x - 2 = 0$.
32. $12x^2 - 8x + 1 = 0$.
33. $8y^2 - 12y + 1 = 0$.
34. $11x^2 - 12x + 1 = 0$.
35. $15x^2 - 10x + 1 = 0$.
36. $12y^2 - 10y + 2 = 0$.
37. $ax^2 - bx - c = 0$.
38. $bx^2 + cx + a = 0$.
39. $a^2y^2 + b^2y + c^2 = 0$.
40. $ax^2 + 2bx + 3c = 0$.
41. $3ax^2 + 2bx + c = 0$.
42. $2ay^2 - by - 2c = 0$.
43. $(x - 1)(x - 2) = 3$.
44. $(x - 1)(x + 2) = 2$.
45. $y(y + 4) = 5$.

46. Solve $s = v_0t - \frac{1}{2}gt^2$ for t; for v_0.

47. Solve $s = \dfrac{n}{2}[2a + (n - 1)d]$ for n.

48. Solve $x^2 + y^2 + 4x + 2y - 1 = 0$ for y.

49. Solve $g = \dfrac{R}{R^2 + x^2}$ for R; for x.

50. Solve $c = \dfrac{b(3kd - 2c)}{3(2kd - c)}$ for c.

9.9. *The graphical solution of a quadratic equation*

Four types of results for quadratic equations are illustrated by the roots of the quadratics (1)–(4) discussed in Art. 9.2. The graphs of the functions of x in the left members of these equations give us a geometric interpretation of their roots. Designating each of these functions as y, we have the four equations: (1) $y = x^2 + 4x - 5$, (2) $y = x^2 + 4x - 1$, (3) $y = x^2 + 4x + 4$, and (4) $y = x^2 + 4x + 8$, which are analyzed in the table below.

Equation	D	Nature of roots	Roots	Graph
(1) $y=x^2+4x-5$	$16+20=36$	Real, unequal, and rational	1 -5	Cuts X-axis at points $(-5, 0)$ and $(1, 0)$
(2) $y=x^2+4x-1$	$16+4=20$	Real, unequal, and irrational	$-2+\sqrt{5}$ $-2-\sqrt{5}$	Cuts X-axis at points $(-2+\sqrt{5}, 0)$ and $(-2-\sqrt{5}, 0)$
(3) $y=x^2+4x+4$	$16-16=0$	Real, equal, and rational	-2 -2	Tangent to X-axis at point $(-2, 0)$
(4) $y=x^2+4x+8$	$16-32=-16$	Imaginary	$-2+2i$ $-2-2i$	Does not touch X-axis

The four graphs are shown in Fig. 14. The curves are all members of the *family of curves*

$$y = x^2 + 4x + c. \tag{5}$$

Each value assigned to c yields a special curve, as indicated in the figure.

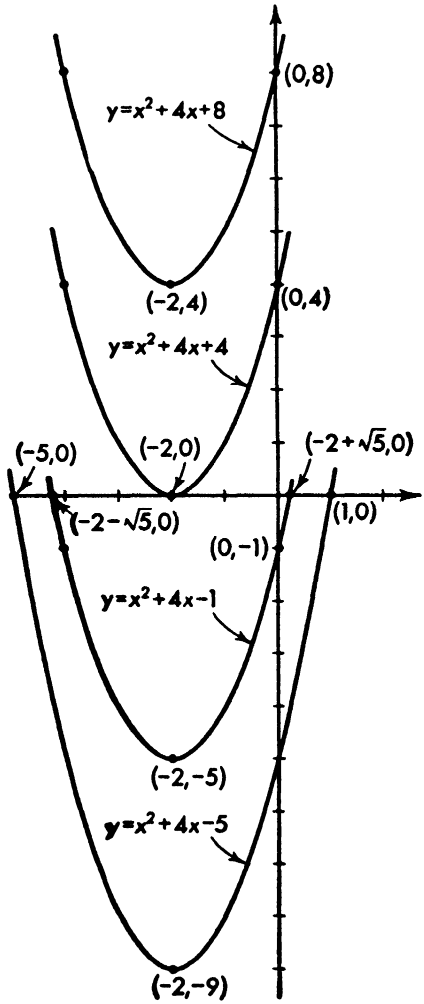

Fig. 14

The graph of (1) crosses the X-axis at the points (1, 0) and $(-5, 0)$. Another way of saying this is that the x-intercepts

of the graph are 1 and -5. These numbers are the roots of the equation

$$x^2 + 4x - 5 = 0. \tag{6}$$

For evidently at $x = 1$ and -5 the curve (1) crosses the X-axis, so that for these values of x, $y = 0$ and hence (6) is satisfied.

Similarly, the graph of (2) crosses the X-axis in two points; but in this case their abscissas are irrational, namely, $-2 + \sqrt{5}$ and $-2 - \sqrt{5}$. These numbers are the roots of the quadratic

$$x^2 + 4x - 1 = 0. \tag{7}$$

In the graph of (3) the curve has in effect been lifted upward until the two x-intercepts have coincided at the point $(-2, 0)$. Hence the equation

$$x^2 + 4x + 4 = 0 \tag{8}$$

may be said still to have two roots (-2 and -2) which are identical. The graph of (3) is said to be *tangent* to the X-axis at $(-2, 0)$.

Finally, since the graph of (4) does not intersect the X-axis, there are no positive or negative values of x for which $y = 0$, and hence there are no real roots of

$$x^2 + 4x + 8 = 0. \tag{9}$$

Summing up, *if the roots of the quadratic*

$$ax^2 + bx + c = 0 \tag{10}$$

are real, we may find them as accurately as the precision of the drawing allows from the graph of the corresponding equation

$$y = ax^2 + bx + c. \tag{11}$$

Incidentally, the graph of (11), where $a \neq 0$, will always be a *parabola*, a very important curve entering into life and mathematics in many ways. Different sets of values for a, b, and c give different parabolas; but all of them are shaped in general like the ones shown in Fig. 10. They open upward

as in that figure if a is positive, and downward when a is negative. The formula for the abscissa (or x-value) of the vertex of each parabola represented by (11) is

$$(12) \qquad x = -\frac{b}{2a}.$$

Example. Sketch the parabola,

$$(13) \qquad y = -2x^2 + 4x - 1.$$

Solution. Using (12) and (13), we find that the coordinates of the vertex are: $x = -\dfrac{4}{-4} = 1$; $y = -2(1^2) + 4 \cdot 1 - 1 = 1$. The curve opens downward, since the coefficient of x^2 is negative. The coordinates, $(0, -1)$, of the point at which the curve crosses the Y-axis are easily found by setting $x = 0$.

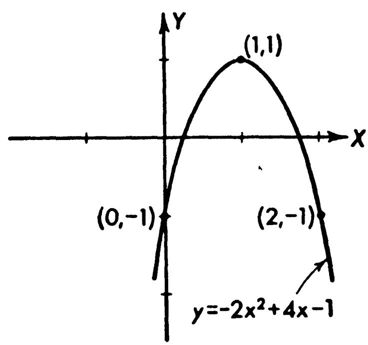

Fig. 15

Since we know the vertex, the direction of the axes, and *one* point on the curve, we can now sketch the parabola shown in Fig. 15. For greater accuracy, we of course use more points whose coordinates satisfy (13). From the graph we might estimate the values of the two roots of

$$(14) \qquad -2x^2 + 4x - 1 = 0$$

as about .3 and 1.7. Actually they are $\dfrac{2 \pm \sqrt{2}}{2}$, or very nearly as estimated.

The graphical solution of a quadratic equation is interesting not only because it gives geometric meaning to the differ-

ent types of roots encountered, but also because it illustrates a very general method of getting approximations to the real roots of an equation in one unknown. This method is very reliable as a last resort if no shorter way can be found. In the case of quadratic equations, however, the algebraic methods are more efficient, precise and practical.

EXERCISE 45

Graph the left members of equations 1–15, *Exercise* 43. *From the graphs, estimate to one decimal place the values of the real roots, and compare with the precise values found algebraically.*

9.10. *Quadratics with given roots*

THEOREM 1. *If r is a root of the equation*

$$ax^2 + bx + c = 0, \tag{1}$$

then $x - r$ is a factor of the left member of (1).

Proof. To say that r is a root of (1) means that

$$ar^2 + br + c = 0. \tag{2}$$

From (1) and (2),

$$(ax^2 + bx + c) - (ar^2 + br + c) = a(x^2 - r^2) + b(x - r),$$

or, since $ar^2 + br + c = 0,$

$$\begin{aligned} ax^2 + bx + c &= a(x - r)(x + r) + b(x - r) \\ &= (x - r)(ax + ar + b). \end{aligned}$$

Example. $2x^2 + 5x - 3 = 0$ has the root $\frac{1}{2}$, and, therefore, by the theorem, $x - \frac{1}{2}$ must be a factor of $2x^2 + 5x - 3$. Actually, $2x^2 + 5x - 3 = 2(x - \frac{1}{2})(x + 3)$.

From Theorem 1 there follows, as a corollary,

THEOREM 2. *All quadratics whose roots are r and s, must take the form*

$$a(x - r)(x - s) = 0, \tag{3}$$

where a can be any constant except zero.

Example 1. Find a quadratic equation with integral coefficients whose roots are $\frac{3}{2}$ and $-\frac{2}{3}$.

Solution. The equation must have the form

$$a(x - \tfrac{3}{2})[x - (-\tfrac{2}{3})] = 0,$$
$$a(x - \tfrac{3}{2})(x + \tfrac{2}{3}) = 0.$$

Letting $a = 6$ to clear fractions, we have

$$6x^2 - 5x - 6 = 0 \text{ (answer).}$$

Example 2. Find a quadratic equation with integral coefficients whose roots are $\dfrac{2 \pm i\sqrt{3}}{2}$.

Solution. The equation must have the form

$$a\left[x - \frac{2 + i\sqrt{3}}{2}\right]\left[x - \frac{2 - i\sqrt{3}}{2}\right] = 0,$$

or, when $a = 4$,

$$(2x - 2 - i\sqrt{3})(2x - 2 + i\sqrt{3}) = 0,$$

or

$$(2x - 2)^2 - (i\sqrt{3})^2 = 0,$$

or

$$4x^2 - 8x + 7 = 0.$$

EXERCISE 46

Find equations with integral coefficients whose roots are the numbers below.

1. $1, 2$. **2.** $3, -1$. **3.** $-1, -3$. **4.** $0, 2$. **5.** $0, -4$. **6.** $\frac{1}{2}, \frac{3}{4}$.
7. $\frac{2}{3}, -\frac{1}{2}$. **8.** $0, \frac{2}{5}$. **9.** $1, -\frac{3}{5}$. **10.** $2 \pm \sqrt{3}$.
11. $3 \pm \sqrt{2}$. **12.** $1 \pm \dfrac{\sqrt{3}}{2}$. **13.** $\dfrac{1 \pm \sqrt{3}}{2}$. **14.** $\pm i$.
15. $1 \pm i$. **16.** $2 \pm 3i$. **17.** $2 \pm i\sqrt{3}$. **18.** $1 \pm \dfrac{i}{2}$.

9.11. *Stated problems leading to quadratic equations*

The solution of many stated problems calls for the use of quadratic equations. If two different numbers satisfy the conditions of the problem, they will be the roots of the

quadratic obtained. If only one number meets the required condition, either this number will appear as a double root or else the second root must be rejected as meaningless. Finally, if the conditions as stated are inconsistent, this fact will appear algebraically in the form of imaginary solutions to the quadratic.

Example 1. Find two consecutive numbers whose product is 56.

Solution. Let $x =$ the smaller number (algebraically).
Then $x + 1 =$ the other number.

$$x(x + 1) = 56.$$

Solving, we have

$$x = 7 \text{ or } -8;$$
$$x + 1 = 8 \text{ or } -7.$$

Hence the answers are: (1), 7 and 8; (2), -8 and -7. Here *both* roots of the quadratic meet the required condition.

Example 2. A building lot has an area of 56 square rods. Its length is one rod greater than its width. Find its dimensions.

Solution. Here the quadratic obtained is the same as that for Example 1; but the root -8 must be rejected as meaningless in this case.

Example 3. The area in square feet of a certain square is 4 units less than its perimeter in feet. Find the length of its side.

Solution. Let $x =$ the length of a side, in feet.
Then $x^2 =$ its area in square feet,
and $4x =$ its perimeter, in feet.

$$x^2 = 4x - 4.$$

Solving, we have

$x = 2$ (answer). Here 2 is a repeated root.

Example 4. The area in square feet of a certain square is 5 units less than its perimeter in feet. Find the length of its side.

Solution. Stating the conditions as in Example 3, we have

$$x^2 = 4x - 5.$$

The quadratic formula yields

$$x = \frac{4 \pm 2i}{2} = 2 \pm i,$$

and hence the conditions are not met by any square.

EXERCISE 47

1. Find two numbers whose difference is 3 and whose product is 40.

2. The base of a ladder 20 feet long which leans against a barn is x feet from the barn. The top of the ladder is $x + 4$ feet above the ground. Find x.

3. A rectangular lot is $\frac{3}{4}$ as wide as it is long. The length of its diagonal is 250 feet. Find its dimensions.

4. A lot is 70 feet longer than it is wide. The length of its diagonal is 130 feet. Find its dimensions.

5. Find two numbers whose sum is 20 and whose product is 99.

6. The length of a rectangle is 3 inches longer than its width. If its length is increased by 2 inches and its width is increased by 3 inches, its area will be doubled. Find its dimensions.

7. The length of a rectangle is twice its width. If its width is increased by 1 inch and its length by 3 inches, its area will be doubled. Find its dimensions.

8. The diagonal of a square is 2 feet longer than its side. Solve for the length of its side (a), by use of a linear equation only; (b), by use of a quadratic equation.

9. The diagonal of a rectangle is 2 inches longer than its length and 9 inches longer than its width. Find its dimensions.

10. The area of a certain square, in square feet, is (a), 5 units more than, (b) equal to, and (c) 5 units less than, its perimeter in

feet. Which of these conditions, if any, are possible, and what are the dimensions in these cases?

11. The length of a rectangle is 1 inch more than its width, and its area in square inches is equal to its perimeter in inches. Find its perimeter if the conditions are consistent.

12. The length of a rectangle is 1 inch more than its width, and its area in square inches is one-half of its perimeter in inches. Find its perimeter if the conditions are consistent.

13. One root of the quadratic, $2x^2 - bx - 1 = 0$, is 1. Find the other root.

14. One root of $3x^2 + 2x + c = 0$ is -1. Find the other root.

15. One root of $ax^2 - 4x + 3 = 0$ is 2. Find the other root.

16. The sum of the roots of the equation $2x^2 - hx + 2k = 0$ is 4, and their product is -3. What must be the values of h and k?

17. Find the value of b if the roots of the equation $3x^2+bx+2=0$ are equal.

Chapter Ten

SPECIAL EQUATIONS IN ONE UNKNOWN

10.1. *The general equation in one unknown*

How can we represent in one algebraic sentence all of the equations in one unknown, say x, which could be written? This appears to be a "large order"; but the answer is simple. It is the equation

$$(1) \qquad f(x) = 0.$$

Thus far we have learned how to solve (1) algebraically when $f(x)$ is a linear function, such as $2x - 3$, or a quadratic function, such as $3x^2 - 2x + 5$.

We have seen further that there is a graphical method of approximating the real roots of (1) (as illustrated in Art. 9.9 for the special case of the quadratic) which is so dependable that *every student should bear it in mind as a last resort when algebraic methods fail him.* The method consists of drawing the graph of the related curve

$$(2) \qquad y = f(x),$$

and then finding, as nearly as the accuracy of the graph allows, the x-intercepts of (2), which will be the real roots of (1).

In this chapter we shall discuss special equations in one unknown for which the *exact* values of the roots can be found by algebraic means. Linear and quadratic equations, which belong in this group, have been disposed of already (Chapters 4 and 9).

10.2. *Solving equations by factoring*

The method of solving a quadratic by factoring the left member when the right member is zero carries over directly to equations of higher degree. The method succeeds when all factors found are linear or quadratic.

Example 1. Solve the equation

$$x^5 - x = 0. \tag{1}$$

Solution. Factoring the left member, we have:

$$\begin{aligned} x^5 - x = x(x^4 - 1) &= x(x^2 + 1)(x^2 - 1) \\ &= x(x^2 + 1)(x - 1)(x + 1). \end{aligned}$$

The roots obtained by setting each of these factors separately equal to zero are 0, i, $-i$, 1, and -1. Since $i^5 = i$ (Art. 8.2), it follows that $i^5 - i = i - i = 0$. The student may check each of the other four roots.

Example 2. Solve the equation

$$x^3 = 1. \tag{2}$$

Solution. Transposing the constant term and factoring the left member, we have: $x^3 - 1 = (x - 1)(x^2 + x + 1) = 0$. When $x-1=0$, $x=1$; when $x^2+x+1=0$, $x=\dfrac{-1\pm i\sqrt{3}}{2}$. Hence the roots of (2), or the three cube roots of 1, are 1, $\dfrac{-1 + i\sqrt{3}}{2}$, and $\dfrac{-1 - i\sqrt{3}}{2}$.

In the last example one might easily overlook the two imaginary roots; but he will not do so if he remembers the following simple and important result, proved in more advanced courses:

THEOREM 1. *Every rational integral equation of the* **n**th *degree has exactly* **n** *roots, not necessarily all different.*

For example, the hundredth degree equation,

$$x(x - 1)^{99} = 0, \tag{3}$$

has exactly 100 roots, including the single root zero and the *multiple* or *repeated* root 1, which is of *multiplicity* 99.

10.3. *Equations with given roots*

As a corollary of the factoring method, and also as an extension of the method of Art. 9.10 applying to quadratics, it is possible to write immediately the equation in one unknown having any given roots.

Example 1. Write an equation whose roots are 0, 1, 2, and -3.

Solution. $a(x - 0)(x - 1)(x - 2)[x - (-3)] = 0$, where a can be any constant. If we let $a = 1$ and perform the indicated multiplication, we have: $x^4 - 7x^2 + 6x = 0$. Another answer is $2x^4 - 14x^2 + 12x = 0$.

Example 2. Write an integral rational equation with integral coefficients whose roots are $1 \pm i$ and $\dfrac{2 \pm \sqrt{3}}{3}$ (four numbers).

Solution. An equation with the desired roots, unsimplified, is

$$a[x - (1 + i)][x - (1 - i)]\left[x - \frac{2 + \sqrt{3}}{3}\right]\left[x - \frac{2 - \sqrt{3}}{3}\right] = 0.$$

This, however, is not an answer because the problem calls for an equation with integral coefficients. Hence we must let $a = 9$ (or 18, or 27, etc.) to clear fractions. With $a = 9$, we have

$$\begin{aligned}(x - 1 - i)(x - 1 + i)&(3x - 2 - \sqrt{3})(3x - 2 + \sqrt{3})\\ &= [(x - 1)^2 - i^2][(3x - 2)^2 - (\sqrt{3})^2]\\ &= (x^2 - 2x + 2)(9x^2 - 12x + 1) = 0,\end{aligned}$$

or $\qquad 9x^4 - 30x^3 + 43x^2 - 26x + 2 = 0.$ (Answer.)

If we had chosen $a = 18$ the equation obtained would have coefficients with the common factor 2. When all common integral factors of the coefficients different from $+1$ or -1

are divided out, the equation is said to be *reduced to simplest form,* as is the answer above.

EXERCISE 48

Solve by the factoring method.

1. $x^3 = -1.$
2. $x^4 = 1.$
3. $x^4 + 2x^2 + 1 = 0.$
4. $x^5 - 2x^3 + x = 0.$
5. $x^3 + x^2 - 6x = 0.$
6. $x^4 - x^3 - 6x^2 = 0.$
7. $x^3 - x^2 - x + 1 = 0.$
8. $x^4 - x^3 + x^2 - x = 0.$
9. $2x^3 - 4x^2 + 3x - 6 = 0.$
10. $x^5 - 2x^4 = x - 2.$
11. $x^4 - 4 = 0.$
12. $x^5 + x^3 + x = 0.$

Find rational integral equations, with integral coefficients and reduced to simplest form, whose roots are the given sets of numbers.

13. 1, 2, 3.
14. 1, -2, -3.
15. 0, 1, -2.
16. 0, 2, -3.
17. ± 1, $\pm i$.
18. 0, $1 \pm \sqrt{2}$.
19. 0, $1 \pm i\sqrt{2}$.
20. 1, $2 \pm 3i$.
21. $\frac{1}{2}$, $1 \pm \frac{i}{2}$.
22. $1 \pm \sqrt{2}$, $1 \pm \frac{\sqrt{3}}{2}$.
23. 0, 1, $-\frac{1}{2}$, $\frac{2}{3}$.
24. 1, 1, 2, 2.
25. $i, i, -i, -i$.
26. 0, 0, 0.
27. $-1, -1, -1$.
28. 0, 0, 1, 1.

10.4. *Equations in quadratic form*

If an equation is quadratic, not in x, but in some function of x such as x^2, x^3, $\frac{1}{x}$, etc., the method for quadratic equations may be used to find numerical values for the function. The solution is then completed by placing the function equal to each of the roots of the quadratic, and solving the two equations thus obtained.

Example 1. Solve the equation

$$x^6 - 7x^3 - 8 = 0. \tag{1}$$

Solution. Writing (1) in the form

$$(x^3)^2 - 7(x^3) - 8 = 0, \tag{2}$$

we see that it is a quadratic in x^3. The solution by factoring or quadratic formula yields

$$x^3 = 8, \tag{3}$$

or

$$x^3 = -1. \tag{4}$$

By the factoring method we find that the roots of (3) are 2 and $-1 \pm i\sqrt{3}$, while those of (4) are -1 and $\dfrac{1 \pm i\sqrt{3}}{2}$. Thus, (1) has six distinct roots.

Example 2. Solve

$$\frac{2x^2 + 3}{x} - \frac{5x}{2x^2 + 3} = 4. \tag{5}$$

Solution. Let $y = \dfrac{2x^2 + 3}{x}$. Then (5) becomes

$$y - \frac{5}{y} = 4, \tag{6}$$

or

$$y^2 - 4y - 5 = 0, \tag{7}$$

from which $y = 5$ or -1. Remembering that it is x whose value is sought, we replace y by the function of x for which it stands, getting

$$\frac{2x^2 + 3}{x} = 5, \tag{8}$$

and

$$\frac{2x^2 + 3}{x} = -1. \tag{9}$$

The roots of (8) are $\dfrac{3}{2}$ and 1; those of (9), $\dfrac{-1 \pm i\sqrt{23}}{4}$. Hence (5) has two real and two imaginary roots. When it is cleared of fractions it is seen to be a fourth degree equation.

EXERCISE 49

Solve the following equations in quadratic form.

1. $2x^4 - x^2 - 6 = 0.$
2. $2x^4 + x^2 - 6 = 0.$
3. $6x^4 + x^2 - 1 = 0.$
4. $6x^4 - x^2 - 1 = 0.$
5. $2x^4 - x^2 - 1 = 0.$
6. $2x^4 + x^2 - 1 = 0.$
7. $x^6 + 7x^3 - 8 = 0.$
8. $x^6 - 7x^3 - 8 = 0.$
9. $x^6 + 26x^3 - 27 = 0.$
10. $x^6 - 26x^3 - 27 = 0.$
11. $8x^6 - 63x^3 - 8 = 0.$
12. $8x^6 + 63x^3 - 8 = 0.$
13. $x^6 - 28x^3 + 27 = 0.$
14. $8x^6 - 19x^3 - 27 = 0.$
15. $8x^6 + 19x^3 - 27 = 0.$
16. $2(x^2+x)^2-5(x^2+x)+3=0.$
17. $2(x^2 - x)^2 - (x^2 - x) - 3 = 0.$
18. $(x^2 - x - 1)^2 - 2x^2 + 2x + 2 = 0.$
19. $(x^2 + x - 1)^2 - 3(x^2 + x - 1) + 2 = 0.$
20. $(x^2 + x + 1)^2 - x^2 - x - 3 = 0.$
21. $(2x^2 - x)^2 + 2x^2 - x - 2 = 0.$
22. $(x^2 + x + 1) + \dfrac{2}{x^2 + x + 1} = 3.$
23. $\dfrac{2(x^2 + 1)}{x} + \dfrac{2x}{x^2 + 1} = 5.$
24. $\dfrac{2(x^2 - 1)}{x} - \dfrac{2x}{x^2 - 1} = 3.$
25. $2(x^2 - x) - \dfrac{3}{x^2 - x} + 1 = 0.$

10.5. *Equations involving radicals*

When the unknown appears in one radicand, or in several, the processes necessary to eliminate the radicals may lead to rational integral equations of the types already considered.

Example 1. Solve the equation

$$\sqrt{2x + 3} + \sqrt{x + 1} = 1 \tag{1}$$

Solution. Here the student often makes the error of lifting off the radicals with the mistaken impression that he is thus squaring both sides. But this is a serious error, since $(a + b)^2$

$= a^2 + 2ab + b^2$, and hence $(\sqrt{2x+3} + \sqrt{x+1})^2 = (\sqrt{2x+3})^2 + 2\sqrt{2x+3}\sqrt{x+1} + (\sqrt{x+1})^2$. Thus the radical $2\sqrt{(2x+3)(x+1)}$ still remains. To get a simpler radicand, however, it is better first to rewrite (1) thus:

$$\sqrt{2x+3} = 1 - \sqrt{x+1}. \tag{2}$$

When its members are squared, (2) becomes

$$2x+3 = 1^2 - 2\sqrt{x+1} + (\sqrt{x+1})^2 = 1 - 2\sqrt{x+1} + x + 1. \tag{3}$$

Simplifying and transposing terms so that the one remaining radical is by itself in the left member, we have

$$2\sqrt{x+1} = -x - 1. \tag{4}$$

A second squaring yields

$$4(x+1) = x^2 + 2x + 1, \tag{5}$$

or

$$x^2 - 2x - 3 = 0, \tag{6}$$

from which $x = 3$ or -1.

But we have not finished the problem. For when we squared both sides in steps (3) and (5) we in effect multiplied both sides by functions of x, and hence may have introduced extraneous roots. Testing $x = 3$ in the left member of (1) we have: $\sqrt{2 \cdot 3 + 3} + \sqrt{3+1} = 3 + 2 = 5$ (not 1), so that 3 is extraneous and must be rejected. But, for $x = -1$, $\sqrt{2(-1)+3} + \sqrt{-1+1} = 1 + 0 = 1$. Thus, -1 satisfies (1) and is the only root.

Example 2. Solve

$$\sqrt{x} = -1. \tag{7}$$

Solution. Squaring both sides, we get $x = 1$; but this must be rejected since $\sqrt{1} \neq -1$. Hence (7) has no root.

Note that (7) is not a rational integral equation in x. Thus while, as noted before, it can be proved that every rational integral equation of the nth degree has n roots, other types of equations may not have any roots.

EXERCISE 50

Solve the following equations involving radicals.

1. $\sqrt{2x-3} = 5.$
2. $\sqrt{x+5} = 5.$
3. $\sqrt{4-x} = 2.$
4. $\sqrt{5x-1} - \sqrt{x-1} = 2.$
5. $\sqrt{x+1} - \sqrt{x-3} = 2.$
6. $\sqrt{1-3x} - \sqrt{3+2x} = 1.$
7. $\sqrt{x+2} + \sqrt{3-x} = 1.$
8. $\sqrt{x+2} + \sqrt{2x+5} = 1.$
9. $\sqrt{2x+3} - \sqrt{4x-1} = 1.$
10. $\sqrt[3]{x+1} = 1.$
11. $\sqrt[3]{x^2-1} = 2.$
12. $\sqrt{x+\sqrt{x-1}} = 1.$
13. $\sqrt{x+\sqrt{x+2}} = 2.$
14. $\sqrt{3x-\sqrt{x^2+3}} = 1.$
15. $\sqrt{3x} + \sqrt{x+1} = \sqrt{8x+1}.$
16. $\sqrt{x+1} + \sqrt{x+2} = \sqrt{2x+3}.$
17. $\sqrt{x+a} + \sqrt{x+b} = \sqrt{2x+a+b}.$
18. $\sqrt{x-1} + \sqrt{x+3} = \sqrt{2x+2}.$

In the following articles a few theorems and rules are given which are found to be of great assistance in solving equations of a higher degree than the second. For supplementary material covering this work a student is referred to the chapter headed "Theory of Equations" usually included in any college algebra text.*

10.6. *The remainder theorem*

The remainder theorem may be stated as follows:

If r *is a constant and if any polynomial in* x *is divided by* $(x - r)$ *until a remainder is obtained that does not contain* x, *that remainder is the value that the polynomial would have if* r *were substituted for* x.†

Proof. Let

$$f(x) = (x - r)Q(x) + R, \tag{1}$$

* For example, *see* Rider's *College Algebra*, alternate edition, pages 187–227.

† A *polynomial in* x, as used here, means an integral and rational function of x, with integral coefficients, such as $3x^3 - 2x^4 + x^2 - 4x + 3$. (*See* Art. 2.1.) For brevity, it is often designated as $f(x)$. Thus, $f(x) = 2x^4 + 3x - 1$ means that $f(x)$ here stands for the particular polynomial, $2x^4 + 3x - 1$.

where $Q(x)$ is the Quotient and R is the constant remainder obtained when $f(x)$ is divided by $x - r$. Since the right member of (1) is the same function of x as the left member, though in different algebraic form, the two sides will be equal when $x = r$. Hence

$$(2) \qquad f(r) = (r - r)Q(r) + R = 0 \cdot Q(r) + R = R.$$

Example.

Divide $2x^3 - 3x^2 + x - 1$ by $x - 1$.

$$(2x^3 - 3x^2 + x - 1) \div (x - 1) = 2x^2 - x + \frac{-1}{x - 1} = 2x^2 - x - \frac{1}{x - 1}$$

$$\begin{array}{rrrr|l} 2x^3 & -3x^2 & +x & -1 & x-1 \\ \cline{5-5} 2x^3 & -2x^2 & & & 2x^2 - x \\ \cline{1-4} & -x^2 & +x & -1 & \\ & -x^2 & +x & & \\ \cline{2-4} & & & -1 & \end{array}$$

In this example $x - r$ is $x - 1$, so that $r = 1$. The remainder, which we shall call R, is -1. Here, according to the theorem, if we substitute 1 for x in the polynomial $2x^3 - 3x^2 + x - 1$, the value of the polynomial will be -1. To check this, we note that

$$2(1)^3 - 3(1)^2 + 1 - 1 = 2 - 3 + 1 - 1 = -1.$$

10.7. *Synthetic division*

A condensation of the operation above which retains only the essential numbers is called *synthetic division.*

Example 1. Divide $2x^3 - 3x^2 + x - 1$ by $x - 1$.

First, copy the coefficients and place 1 (the value of r), in the position of a divisor, as in the first line below.

$$\begin{array}{rrrr|l} 2 & -3 & 1 & -1 & 1 \\ & 2 & -1 & 0 & \\ \hline 2 & -1 & 0 & -1 & \end{array}$$

Next, draw a line two spaces below the coefficients and copy the first number below this line. Multiply this number

(2) by the number (1) in the divisor's position, place the result (2) under the next coefficient (-3) and add. Place the result (-1) under the line. Multiply this number (-1) by the number in the divisor's position (1), place result (-1) below the next coefficient (1), and add. Place the result (0) below the line. Multiply this number (0) by the number (1) in the divisor's position, place the result (0) below the next coefficient (-1), and add. Place the result (-1) below the line. The first three numbers below the line are the coefficients of the Quotient and the last number (-1) below the line is R, the remainder. The degree of the Quotient will be one less than the degree of the polynomial divided. The complete quotient, then, is $2x^2 - x - \dfrac{1}{x - 1}$.

Example 2. Divide $3x^3 - 4x^2 - 5x + 7$ by $x - 2$.

Solution.

$$\begin{array}{rrrr|l} 3 & -4 & -5 & 7 & \underline{2} \\ & 6 & 4 & -2 & \\ \hline 3 & 2 & -1 & 5 & \end{array}$$

The Quotient is $3x^2 + 2x - 1$ and the remainder (R) is 5.

The complete quotient is $3x^2 + 2x - 1 + \dfrac{5}{x - 2}$.

For our purpose, the remainder (5) is the most important part of the result obtained because it is the value of the polynomial with 2 substituted for x.

This is stated in functional notation as follows:

If $\qquad f(x) = 3x^3 - 4x^2 - 5x + 7,$

then $\qquad f(2) = 5.$

Example 3. Find $f(2)$ by synthetic division if

$$f(x) = 5x^4 - 3x^3 + 6x^2 + x - 4.$$

Solution.

$$\begin{array}{rrrrr|l} 5 & -3 & 6 & 1 & -4 & \underline{2} \\ & 10 & 14 & 40 & 82 & \\ \hline 5 & 7 & 20 & 41 & 78 & \end{array}$$

$$\therefore f(2) = 78.$$

In the examples given, every power of x from the highest power down to the constant occurs. That is: x^3, x^2, x and the constant term all occur. If the constant or any power of x is missing, a zero must be placed in its proper position in writing the coefficients in form for synthetic division.

Example 4. Divide $2x^5 - 2x^3 + 1$ by $(x + 3)$.

Solution. To determine the number to place in the divisor's position, we let $x - r = x + 3$, the divisor, and solve for r, getting $r = -3$.

Notice that the x^4 term, the x^2 term, and the x term are missing from the expression. We must use zero for the coefficient of each of these.

$$\begin{array}{rrrrrr|l} 2 & 0 & -2 & 0 & 0 & 1 & \underline{-3} \\ & -6 & 18 & -48 & 144 & -432 & \\ \hline 2 & -6 & 16 & -48 & 144 & -431 & \end{array}$$

Thus, if
$$f(x) = 2x^5 - 2x^3 + 1,$$
$$f(-3) = -431.$$

10.8. *The factor theorem*

The factor theorem is usually stated as follows:

If $f(r) = 0$, then $(x - r)$ is a factor of $f(x)$.

We know that $f(r) = R$, the remainder. Thus, if $f(r) = 0$, then $R = 0$, and hence $(x - r)$, divides the function exactly. In other words, $x - r$ is a factor of the function. Synthetic division is used to determine whether or not $f(r) = 0$.

Example 5. Is $x - 2$ a factor of

$$f(x) = 2x^3 - 3x^2 + 5x - 14?$$

Solution.

$$\begin{array}{rrrr|l} 2 & -3 & 5 & -14 & \underline{2} \\ & 4 & 2 & 14 & \\ \hline 2 & 1 & 7 & 0 & \end{array}$$

$\therefore f(2) = 0$, and $x - 2$ is a factor of $f(x)$.

EXERCISE 51

Divide, using synthetic division.

1. $(3x^3 - 2x^2 + 4x - 5) \div (x - 1)$.
2. $(5x^4 + 3x^3 - x^2 - x - 2) \div (x + 1)$.
3. $(2x^4 - 7x^3 + 8x^2 - 7x + 6) \div (x - 2)$.
4. $(3x^4 + 6x^3 - 2x^2 - 3x + 2) \div (x + 2)$.
5. $(4x^3 + 12x^2 - 3x - 9) \div (x + 3)$.
6. $(x^5 - 5x^4 - x + 5) \div (x - 5)$.
7. $(3x^4 - 12x^3 - 8x + 32) \div (x - 4)$.
8. $(2x^4 - 12x^3 - 3x + 18) \div (x - 6)$.
9. $(3x^3 - 5x^2 + 2x + 4) \div (x - 3)$.
10. $(5x^3 + 4x^2 - 3) \div (x - 2)$.
11. $(3x^4 - 2x^2 + 5) \div (x - 1)$.
12. $(4x^5 - 3x^3 + 2x^2 - 5) \div (x + 1)$.
13. $(7x^3 - 6x^2 + 5x) \div (x + 4)$.
14. $(3x^4 - 6x^3 - 2x^2 + 4x) \div (x - 2)$.

By use of the factor theorem determine whether or not the expression on the left is a factor of the polynomial on the right.

15. $x + 2;\ x^4 + 2x^3 - x - 2$.
16. $x - 1;\ x^4 + 2x^3 - x - 2$.
17. $x + 1;\ 3x^4 + x^3 + x^2 + 4x + 1$.
18. $x + 3;\ 2x^4 + 6x^3 - x - 3$.
19. $x + 2;\ 3x^3 + x^2 - 10x$.
20. $x - 3;\ 5x^4 - 3x^3 + x - 2$.
21. $x + 4;\ x^3 - 3x^2 + 2x - 8$.

10.9. *Theorem on rational roots*

If an equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

with integral coefficients, has a rational root $\frac{c}{d}$, *where* $\frac{c}{d}$ *is reduced to lowest terms, then* $\mathbf{c}$ *is a divisor of* $\mathbf{a_n}$ *and* $\mathbf{d}$ *is a divisor of* $\mathbf{a_0}$.

Example. If the equation

$$3x^3 - 4x^2 + 5x + 2 = 0$$

has a rational root, it must be one of the following numbers.

$$\pm \tfrac{2}{3}, \quad \pm \tfrac{2}{1}, \quad \pm \tfrac{1}{3}, \quad \pm \tfrac{1}{1}.$$

Corollary. Any rational root of an equation

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n = 0,$$

with integral coefficients, is an integral divisor of b_n.

Example. If the equation

$$x^3 - 3x^2 + 4x + 12 = 0$$

has a rational root, it will be one of the following numbers.

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 4, \quad \pm 6, \quad \pm 12.$$

10.10. *Upper and lower limits of roots*

The possible rational roots are checked by synthetic division to determine whether or not they are actual roots. If, when any positive number is tested, the sums beneath the line are all positive, or zero, it can be proved * that there is no root larger than the number being tested. This number is therefore an upper limit of the roots.

Similarly if, when a negative number is tested, the sums are alternately plus and minus throughout the line, it can be shown that there is no root less than the number being tested. That number is therefore a lower limit.

Example. Test the number 2 as a possible root of

$$3x^3 - 4x^2 + 5x + 2 = 0.$$

$$\begin{array}{rrrr|l} 3 & -4 & 5 & 2 & \underline{2} \\ & 6 & 4 & 18 & \\ \hline 3 & 2 & 9 & 20 & \end{array}$$

* The proof is not difficult. The student is challenged to try it in problem 23 of Exercise 51. See also problem 24.

The signs are all plus in the lower line and hence the roots of this equation are all less than 2.

Test -1 as a possible root.

$$\begin{array}{rrrr|l} 3 & -4 & 5 & 2 & \underline{-1} \\ & -3 & 7 & -12 & \\ \hline 3 & -7 & 12 & -10 & \end{array}$$

The signs in the lower line are alternately plus and minus, and hence the roots of the equation are greater than -1.

By this means we may eliminate some of the "possible roots" without testing them by synthetic division.

10.11. *Depressed equations*

Consider the equation

(1) $$(x^2 - 3x + 2)(x - 3) = 0.$$

Any value of x that makes $x - 3 = 0$, or any value of x that makes $x^2 - 3x + 2 = 0$, is a root of (1). The statements, $x - 3 = 0$, and $x^2 - 3x + 2 = 0$ are called *depressed equations* with relation to equation (1). The depressed equation is found by synthetic division.

Example 1. Solve: $x^3 - 6x^2 + 11x - 6 = 0.$

Solution.

$$\begin{array}{rrrr|l} 1 & -6 & 11 & -6 & \underline{3} \\ & 3 & 9 & 6 & \\ \hline 1 & -3 & 2 & 0 & \end{array}$$

Here the remainder is zero. Use the numbers below the line as coefficients and reduce the degree of the expression by one. The depressed equation, then, is $x^2 - 3x + 2 = 0$. Since any root of the depressed equation is also a root of the original equation, we solve the depressed equation for the remaining roots.

$$x^2 - 3x + 2 = 0,$$

or $$(x - 2)(x - 1) = 0.$$

$$x - 2 = 0; \quad x - 1 = 0;$$

$$x = 2; \qquad x = 1.$$

But our synthetic division shows that 3 is a root, since it produces a zero remainder. Therefore, the roots of the original equation are 3, 2, and 1.

Example 2. Find the roots of the equation

$$4x^3 + 8x^2 - 3x - 6 = 0.$$

Solution. The possible rational roots are:

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm \tfrac{1}{2}, \quad \pm \tfrac{3}{2}, \quad \pm \tfrac{1}{4}, \quad \pm \tfrac{3}{4}.$$

Testing -3, we have

$$\begin{array}{rrrr|l} 4 & 8 & -3 & -6 & \underline{-3} \\ & -12 & 12 & -27 & \\ \hline 4 & -4 & 9 & -33 & \end{array}$$

Thus, -3 is not a root, and this test shows that all roots are greater than -3, since the sums below the line alternate in sign. Next,

$$\begin{array}{rrrr|l} 4 & 8 & -3 & -6 & \underline{-2} \\ & -8 & 0 & 6 & \\ \hline 4 & 0 & -3 & 0 & \end{array}$$

This test shows that -2 is a root, since the remainder is zero. The depressed equation is $4x^2 + 0x - 3 = 0$. Upon solving the latter, we find that $x^2 = \frac{3}{4}$, and $x = \pm\sqrt{\frac{3}{4}} = \pm\frac{1}{2}\sqrt{3}$. Thus the roots are -2, $\dfrac{\sqrt{3}}{2}$, and $-\dfrac{\sqrt{3}}{2}$.

EXERCISE 52

By use of the preceding theory, determine what numbers could possibly be roots of the following equations, and then test them. If the final depressed equation is quadratic, find all the roots.

1. $x^3 - 3x^2 - x + 3 = 0.$ **2.** $x^3 - 3x + 2 = 0.$

3. $x^3 + 2x - 3 = 0.$ **4.** $x^4 - 2x^3 + 1 = 0.$

5. $x^4 - 3x^2 + 2 = 0.$ **6.** $2x^5 + 3x^4 - 3x^2 + 2 = 0.$

7. $2x^3 - x^2 - 2x + 1 = 0.$ **8.** $6x^3 + 19x^2 + 15x + 2 = 0.$

9. $x^4 + 3x^3 - 3x^2 - 12x - 4 = 0.$

10. $6x^3 + 19x^2 + x - 6 = 0.$

11. $x^4 - 2x^3 - 6x^2 + 6x + 9 = 0.$

12. $8x^4 - 4x^3 - 14x^2 + 5x + 5 = 0.$

13. $6x^4 + 3x^3 - 11x^2 - 4x + 4 = 0.$

14. $2x^4 + 5x^2 + 2 = 0.$

15. $3x^4 + 2x^3 + 13x^2 + 8x + 4 = 0.$

16. $x^3 + x - 2 = 0.$

17. $3x^4 + 2x^3 - 49x^2 - 32x + 16 = 0.$

18. $x^5 + 3x^3 - x^2 - 3 = 0.$

19. $x^5 - 4x^3 - 8x^2 + 32 = 0.$

20. $4x^4 - 12x^3 - 25x^2 + 27x + 36 = 0.$

21. $x^4 + 8x^3 - 12x^2 - 24x + 27 = 0.$

22. $x^4 + 3x^3 + x^2 - 2 = 0.$

23. Prove the statement beginning ". . . it can be proved" in the first paragraph of Art. 10.10.

24. Prove the statement beginning ". . . it can be shown" in the second paragraph of Art. 10.10.

Chapter Eleven

SIMULTANEOUS EQUATIONS

11.1. *The general problem*

In this chapter we shall deal with pairs of simultaneous equations in two unknowns, at least one member of each pair being quadratic, or of the second degree, in one or both of the unknowns. Equation-pairs of this sort arise frequently in the solution of simple problems.

As in the case of linear equations, the solution of simultaneous quadratics may be obtained graphically by use of the rectangular coordinate system. In this case, however, it is necessary to plot loci which are not lines. Some typical ones will be discussed in the next article.

11.2. *Typical loci*

A quadratic equation in two variables may not have a locus. For example, there is no pair of values for x and y which satisfies the equation

$$x^2 + y^2 = -1, \tag{1}$$

since the sum of two squares of real numbers cannot be negative. But if it exists at all the locus is always one of five things: (a), an ellipse (Fig. 16), including the circle as a special case; (b), a parabola (Fig. 17); (c), a hyperbola (Fig. 18); (d), two straight lines which may or may not coincide (Fig. 19), or (e), a single point. These curves are studied in *analytic geometry*. For our purposes we may note simply that if the student has in mind the general appear-

ance of each of the three main curves, he can usually sketch one in roughly when he has located a few points on it.

Example. Sketch the curve

$$(2) \qquad 9x^2 + 25y^2 = 225.$$

Solution. Solving (2) for y, we have

$$(3) \qquad y = \pm \tfrac{3}{5}\sqrt{25 - x^2}.$$

Assigning to x the values 0, 3, 5, -3, and -5, we find from (3) that the corresponding values of y are ± 3, $\pm \frac{12}{5}$, 0, $\pm \frac{12}{5}$, and 0. These points are on the ellipse of Fig. 16. Note that

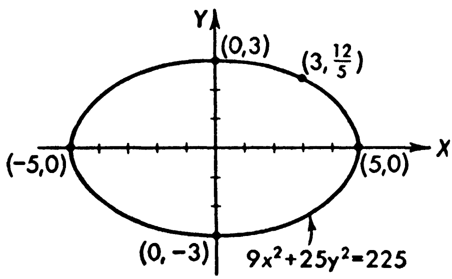

Fig. 16. Ellipse

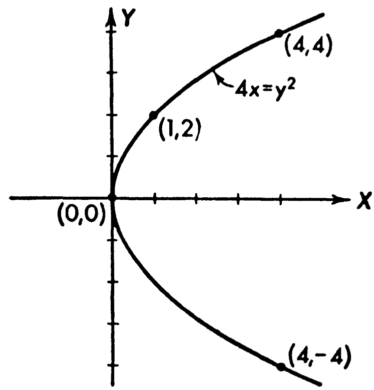

Fig. 17. Parabola

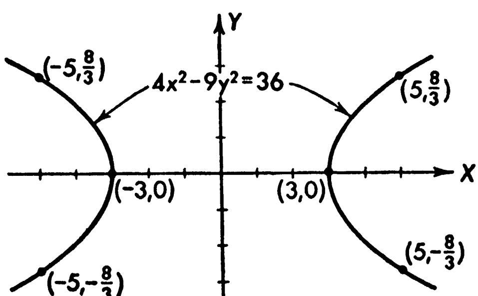

Fig. 18. Hyperbola

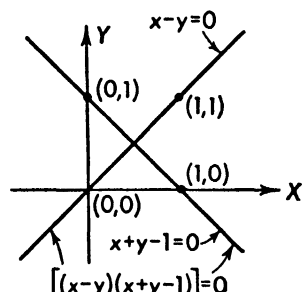

Fig. 19. Two straight lines

if x exceeds 5 numerically, y is imaginary. This means that the curve does not extend to the right of the line $x = 5$, nor to the left of the line $x = -5$.

EXERCISE 53

The graphs of the equations in problems 1 and 2 below are ellipses; in 3 and 4, parabolas; in 5 and 6, hyperbolas. Find at least 6 points on each, and sketch the curve.

1. $x^2 + y^2 = 25.$
2. $4x^2 + 9y^2 = 36.$
3. $y = 2x^2 + x - 4.$
4. $y^2 - 4y - x + 3 = 0.$
5. $x^2 - y^2 = 9.$
6. $9y^2 - 4x^2 = 36.$

Draw the graphs indicated for problems 7–15. They are straight lines.

7. $(x - y)(x + y - 1) = 0.$
8. $(3x-4y-6)(4x+3y-6)=0.$
9. $(x - y)^2 = 1.$
10. $(x - y - 2)^2 = 0.$
11. $x^2 - 4 = 0.$
12. $y^2 = 0.$
13. $y^2 = 4.$
14. $(2x + 3y + 6)^2 = 0.$
15. $y^2 - x^2 = 0.$

Graph the following equations.

16. $9x^2 + 4y^2 = 36.$
17. $9x^2 - 4y^2 = 36.$
18. $9x^2 - 4y^2 = 0.$
19. $x^2 + y^2 = 9.$
20. $x^2 - y^2 = 9.$
21. $x^2 - y^2 = 0.$
22. $(x - y)^2 = 0.$
23. $x^2 + y^2 = 0.$
24. $(x - 2)^2 + (y + 1)^2 = 0.$

11.3. *Linear and quadratic equations*

Suppose, for example, we seek the dimensions of a rectangle whose diagonal is 10 inches long and whose perimeter is 28 inches.

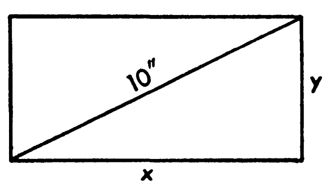

Fig. 20

Designating the two unknowns, the measures of length and width in inches, by x and y respectively, we have, from Fig. 20 and the Pythagorean relation,

(1) $$x^2 + y^2 = 10^2.$$

Also, since half of the perimeter is 14,

(2) $$x + y = 14.$$

Solving (2) for y, we have

(3) $$y = 14 - x.$$

Substitution of the value of y from (3) in (1) yields

(4) $$x^2 + (14 - x)^2 = 100.$$

The solutions of (4) are $x = 6$ or 8. Substituting in (2), or, better yet, in (3), we get the solutions: $x = 6$, $y = 8$, and $x = 8$, $y = 6$. Both of these pairs satisfy (1) and (2); but since algebraic solutions must be examined in the light of the demands of the problem, and since by agreement x stands for the measure of length, the first pair of values must be rejected. The required rectangle is 8 inches long and 6 inches wide.

To solve a linear and quadratic pair, then, we proceed as follows, as illustrated by the example above.

1. Solve the linear equation for one letter in terms of the other.
2. Substitute the obtained literal value of the first letter in the second degree equation.
3. Solve the resulting quadratic in the second letter.
4. Substitute each of the two quadratic roots now found in the *linear* equation to find the corresponding value of the first letter.

Consider next the graphical interpretation of the above problem. The circle (1) and the line (2) intersect at the points (8, 6) and (6, 8). (Fig. 21.) It should be noted that there are *two* points on the circle and only *one* on the line for which $x = 8$. This suggests the reason for the emphasis upon substitution of the found value in the *linear* instead of the quadratic equation.

Suppose, in this problem, that the perimeter were increased to $20\sqrt{2}$ inches, with the diagonal length unchanged. The algebraic solution of the two equations

$$x^2 + y^2 = 100, \tag{1}$$

and

$$x + y = 10\sqrt{2}, \tag{5}$$

yields the one pair of solutions, $x = 5\sqrt{2}$, $y = 5\sqrt{2}$. Geometrically, this means that the line (5) touches the circle (1) at the single point $(5\sqrt{2}, 5\sqrt{2})$ (Fig. 21). Finally, with an

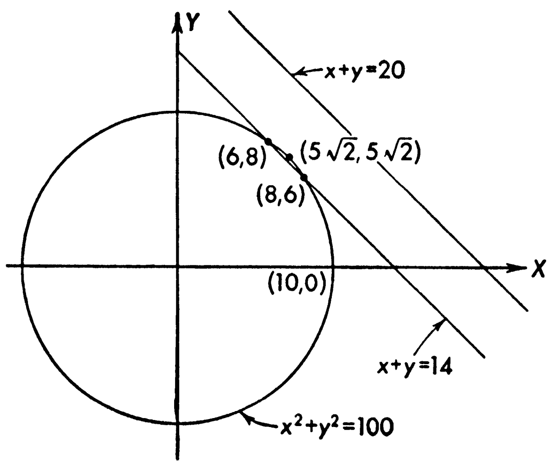

Fig. 21

assumed perimeter of 40, the algebraic solution is imaginary, as might be suspected from the fact that the line, $x + y = 20$, in Fig. 21, does not cross the circle.

In general, the physical impossibility of a pair of simultaneous conditions is indicated *algebraically* whenever i appears in the solution of the equations stating the conditions, as well as by inconsistent equations; and it is shown *graphically* by the nonintersection of the loci of these equations. This result applies not only to linear and quadratic pairs, but also to simultaneous equations of many types and degrees in two unknowns.

11.4. *The number of solution-pairs*

An inspection of the sample second degree curves shown in Figs. 16–19 indicates that a straight line cannot cross one of them in more than two points, and also that two of them cannot intersect in more than four points. The corresponding algebraic results may be stated as follows:

A.* *There are at most two distinct pairs of real numbers which satisfy simultaneously a linear and quadratic equation in two unknowns.*

B.* *There are at most four distinct pairs of real numbers which satisfy simultaneously two quadratic equations in two unknowns.*

EXERCISE 54

Solve the following equations graphically, estimating the coordinates of the points of intersection in case they intersect. Then check by solving algebraically.

1. $x^2 + y^2 = 25,$ $3x - 4y = 0.$
2. $x^2 + y^2 = 25,$ $x + y = 5.$
3. $x^2 + y^2 = 25,$ $x - y = 0.$
4. $x^2 + y^2 = 25,$ $3y - 2x = 1.$
5. $x^2 + y^2 = 25,$ $3x - 4y = 25.$
6. $x^2 + y^2 = 25,$ $y = 6.$
7. $x^2 - y^2 = 16,$ $3x - 5y = 0.$
8. $x^2 - y^2 = 16,$ $y - 3 = 0.$
9. $x^2 - y^2 = 16,$ $x + y = 4.$
10. $x^2 - y^2 = 16,$ $2x - 3y = 1.$
11. $x^2 - y^2 = 16,$ $x = 3.$
12. $x^2 - y = 0,$ $3y - 4x = 4.$
13. $x^2 - y = 0,$ $2x - y = 1.$
14. $x^2 - y = 0,$ $2x - y = 2.$
15. $x - y^2 = 0,$ $x + y = 2.$
16. $x - y^2 = 0,$ $4 - x = 0.$

* It is assumed here that the equations are *independent*. That is, we bar pairs, such as $x + y - 1 = 0$ and $(x + y - 1)(x - y) = 0$, whose graphs have a straight line in common, or such as $x^2 + y^2 = 1$ and $2x^2 + 2y^2 = 2$, whose graphs are identical.

Solve algebraically.

17. $x^2 + xy + y^2 = 3,$
$x + y = 2.$

18. $x^2 + 4y^2 = 2,$
$x + 2y = 0.$

19. $4y^2+xy+x+2y-1=0,$
$x-4y=4.$

20. $(x^2 - xy - y^2) = 1,$
$(x + y) = 1.$

21. $xy = ab,$
$2x + y = 2a + b.$

22. $2x - y = a + 2b,$
$xy + ab = 0.$

23. Find the dimensions of a rectangular field whose area is 20 square rods and whose perimeter is 24 rods.

24. Find the dimensions of a rectangle whose perimeter is 34 inches and whose diagonal is 13 inches.

11.5. *Simultaneous quadratics in general*

The algebraic solution of simultaneous quadratics in two unknowns is in the general case long and tedious, involving the solution of a fourth degree equation. We shall consider, in the next two articles, two special but important cases.

11.6. *Simultaneous equations in linear form*

If no more than two of the five quantities, x^2, y^2, xy, x, and y appear altogether in two simultaneous quadratics, the method for *linear* equations may be applied at once to get these two unknowns. After the latter are found, the values of x and y follow at once. Each of the following seven pairings is of interest here: x^2 and y^2; x^2 and xy; y^2 and xy; x^2 and y; y^2 and x; xy and x; xy and y.

Example 1. Solve simultaneously.

$$(1) \qquad x^2 + y^2 = 13,$$
$$(2) \qquad x^2 - y^2 = -5.$$

Solution. Solving first for x^2 and y^2, we get $x^2 = 4$ and $y^2 = 9$. Hence $x = \pm 2$, $y = \pm 3$. The solution is not com-

plete, however, until the values have been *paired* properly. Since $y = \pm 3$ when $x = 2$ and also when $x = -2$, the pairs are $(2, 3)$, $(2, -3)$, $(-2, 3)$, and $(-2, -3)$ (Fig. 22).

Example 2. Solve simultaneously.

$$2x^2 - 3xy = 1, \tag{3}$$
$$3x^2 + xy = 7. \tag{4}$$

Solution. Treating (3) and (4) as linear in the unknowns x^2 and xy, we find that $x^2 = 2$ and $xy = 1$. Hence $x = \pm\sqrt{2}$, $y = \frac{1}{x} = \pm\sqrt{\frac{1}{2}}$ or $\pm\frac{\sqrt{2}}{2}$, and the solutions are $\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right)$.

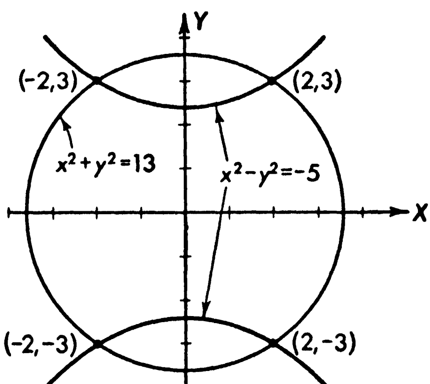

Fig. 22

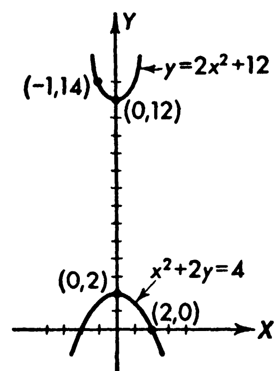

Fig. 23

Example 3. Solve simultaneously.

$$x^2 + 2y = 4, \tag{5}$$
$$2x^2 - y = -12. \tag{6}$$

Solution. These equations are linear in x^2 and y. Solving, we get $x^2 = -4$, $y = 4$; so that the final solutions are $(2i, 4)$ and $(-2i, 4)$. The imaginary values for x indicate that the curves do not intersect. This is seen in Fig. 23 to be the case.

EXERCISE 55

Solve algebraically. In problems 1–6, *solve graphically also.*

1. $x^2 + y^2 = 25,$
$x^2 - y^2 = 4.$

2. $x^2 + y^2 = 25,$
$2x^2 + 3y^2 = 50.$

3. $x^2 + y^2 = 1,$
$x^2 - y^2 = 4.$

4. $x^2 + y^2 = 4,$
$4x^2 + 9y^2 = 36.$

5. $2x - 3y^2 = 1,$
$3x + 2y^2 = 8.$

6. $4x^2 - y = 0,$
$3x^2 - 1 - 2y = 0.$

7. $2x^2 + xy = 6,$
$x^2 + 2xy = 0.$

8. $3xy + y^2 = 1,$
$4xy + y^2 = 2.$

9. $4x^2 + 3y = 5,$
$3y - 8x^2 + 4 = 0.$

10. $3y^2 + 2x - 1 = 0,$
$2y^2 - 3x + 8 = 0.$

11. $3(xy - x) = 1,$
$3x(y + 2) = 7.$

12. $y(2x - 3) = 3,$
$2y - xy + 1 = 0.$

13. $x^2 = 4xy - 3,$
$2xy = 3x^2 + 4.$

14. $y^2 = 3xy - 5,$
$6xy + 2 + y^2 = 0.$

15. $2x^2 + 4y^2 = 3a,$
$3x^2 - 8y^2 = -a.$

16. $2x^2 + 3xy = 2a + 3b,$
$3x^2 - 2xy = 3a - 2b.$

11.7. *Equations reducible to simpler forms*

When quadratics in two unknowns are written with the right members zero, the simultaneous solution of a pair of them is much simplified if the left member of at least one of the equations is or can be factored.

Example 1. Solve simultaneously:

$$(3x - 2y - 1)(x + y + 1) = 0, \tag{1}$$

$$x^2 + y^2 - 25 = 0. \tag{2}$$

Solution. Since any pair of values satisfying

$$3x - 2y - 1 = 0, \tag{3}$$

or

$$x + y + 1 = 0, \tag{4}$$

will also satisfy (1), it follows that any common solution of (3) and (2) or of (4) and (2) will be one of the desired solu-

tions of (1) and (2). Hence the problem reduces to the solution of two linear-and-quadratic pairs: (3)–(2) and (4)–(2). The solutions of (3) and (2) are (3, 4) and $(-\frac{33}{13}, -\frac{56}{13})$; those of (4) and (2) are $(-4, 3)$ and $(3, -4)$. The loci of (1) and (2) are shown in Fig. 24.

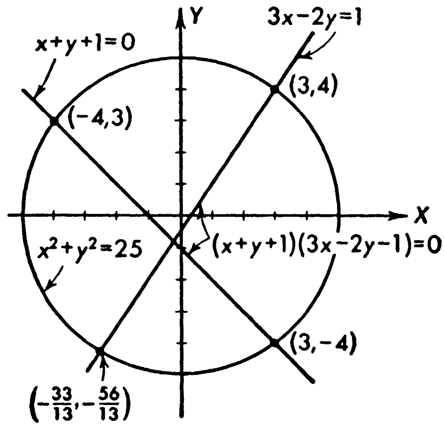

Fig. 24

Example 2. Solve simultaneously:

$$(5) \qquad (2x - 3y - 4)(x + y - 2) = 0,$$
$$(6) \qquad (3x - y + 1)(2x - y - 3) = 0.$$

Solution. Equation (5) is satisfied by the coordinates of any point on either one of the lines

$$(7) \qquad 2x - 3y - 4 = 0,$$

and

$$(8) \qquad x + y - 2 = 0.$$

Similarly, (6) is satisfied by solutions of either

$$(9) \qquad 3x - y + 1 = 0,$$

or

$$(10) \qquad 2x - y - 3 = 0.$$

Hence, the problem reduces to the simultaneous solution of the following four pairs of linear equations: (7)–(9), (7)–(10),

(8)–(9), and (8)–(10). We must be sure that each pair comprises a factor from (5) and also a factor from (6). The rest of the solution is left to the student.

Evidently the method of Example 1 could have been used if the left member of either (5) or (6) had been left in the unfactored form. Here the process is shortened.

Example 3. Solve simultaneously:

$$x^2 - xy = 3, \tag{11}$$

$$4y^2 - 3xy = 2. \tag{12}$$

Solution. This system does not come directly under the types illustrated in Examples 1 and 2. However, it can be reduced to such a type by first producing an equation from the given pair in which the constant term is zero. This new equation is then used to form a system which may be solved in the same manner as the above examples.

Both members of (11) and (12) are multiplied respectively by 2 and 3, resulting in the equations:

$$2x^2 - 2xy = 6, \tag{13}$$

$$12y^2 - 9xy = -6. \tag{14}$$

Adding corresponding members of (13) and (14), we get

$$2x^2 - 11xy + 12y^2 = 0. \tag{15}$$

Since all common solutions of (11) and (12) must also satisfy (15), a new system can be established in which (15) is used as one of the equations, with either (11) or (12) as the other. The simpler of the two — in this case, (11) — should be given preference.

The left member of (15) has the factors $2x - 3y$ and $x - 4y$, so that the new system becomes:

$$x^2 - xy = 3, \tag{16}$$

$$(2x - 3y)(x - 4y) = 0. \tag{17}$$

This system presents no new difficulty, and by the method of Example 1 the solutions are found to be $(3, 2)(-3, -2)$ $(2, \frac{1}{2})$ and $(-2, -\frac{1}{2})$.

EXERCISE 56

Solve algebraically. In problems 1–6, *solve graphically also.*

1. $x^2 + y^2 = 25,$
$(x - 3)(x + y + 1) = 0.$

2. $x^2 - 4y = 0,$
$(x - y)(2x + 4y + 1) = 0.$

3. $x^2 - y^2 = 16,$
$(3x - 5y)(y - 3) = 0.$

4. $x^2 - y^2 = 16,$
$(x - y - 2)(x - 2y + 1) = 0.$

5. $x^2 + y^2 = 25,$
$(x - y)(x + y) = 0.$

6. $x^2 - y^2 = 16,$
$(x - y - 1)(x + y - 1) = 0.$

7. $x^2 + xy + 2y^2 = 1,$
$(x - y)(x + y) = 0.$

8. $2x^2 + 2xy - y^2 = 2,$
$(2x - y)(x + y - 3) = 0.$

9. $3x^2 - xy - y^2 = 1,$
$(x + y - 2)(2x - y - 1) = 0.$

10. $x^2 - 2xy + 4y^2 = 1,$
$(x - 2y)(x + 4y - 3) = 0.$

Factor both left members, and then solve algebraically.

11. $2x^2 + xy - y^2 = 0,$
$x^2 - 4y^2 = 0.$

12. $6x^2 - 7xy - 3y^2 = 0,$
$(2x - y)^2 - 4 = 0.$

13. $(x + y - 1)^2 - 9 = 0,$
$(2x + y)^2 - 4 = 0.$

14. $2x^2 + 5xy - 3y^2 = 0,$
$(x + y)^2 - 1 = 0.$

Solve by the method of illustrative Example 3.

15. $x^2 - 3y^2 = -2,$
$xy + 2y^2 = 3.$

16. $3x^2 + xy = 5,$
$x^2 + y^2 = 5.$

17. $2x^2 - 13xy + 2y^2 = -9,$
$x^2 + 3xy + y^2 - 5 = 0.$

18. $2y^2 + 3xy - 4 = 0,$
$3x^2 + 2xy - 2 = 0.$

Chapter Twelve

RATIOS, PROPORTIONS, AND VARIATIONS

12.1. *Ratios*

A *ratio* is a fraction which compares two things in terms of the same unit. Its value is determined by the relative sizes of the things compared, and *not* by the unit chosen. Thus the ratio of 6 inches to 3 feet is $\frac{1}{6}$, since $\frac{6 \text{ (inches)}}{36 \text{ (inches)}} = \frac{1}{6}$, and also $\frac{\frac{1}{2} \text{ (foot)}}{3 \text{ (foot)}} = \frac{1}{6}$.

Ratios are of great importance in the branch of mathematics known as *trigonometry*. One of its uses is to measure the distance to an inaccessible object, such as the top of a mountain or a spot on the moon, by equating the ratios of corresponding sides of similar triangles.

12.2. *Proportions*

A proportion is an equation whose two members are ratios. Thus, to get the height, CD, of a cliff, a sight is taken at O (Fig. 25), and the lengths, $OA = 5$ feet, $AB = 3$ feet, and

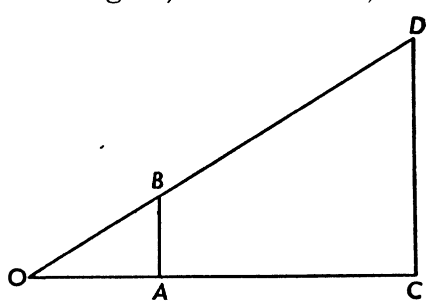

Fig. 25

$OC = 200$ yards, are measured directly. (The representation in Fig. 25 shows OA and AB too large in comparison with OC and CD. Such a figure is called a *diagram* rather than a *scale drawing.*) Then, from similar triangles,

$$(1) \qquad \frac{CD \text{ (in yards)}}{200 \text{ (yards)}} = \frac{3 \text{ (feet)}}{5 \text{ (feet)}} = \frac{3}{5}.^*$$

From (1) it follows that

$$(2) \qquad CD = (\tfrac{3}{5})(200 \text{ yards}) = 120 \text{ yards},$$

the height of the not-necessarily-climbed cliff.

The proportion

$$(3) \qquad \frac{a}{b} = \frac{c}{d}$$

is sometimes written

$$(4) \qquad a : b = c : d,$$

and is read, "a is to b as c is to d." Here b and c are called *means*, and a and d are *extremes*. There would, however, be no point in the new symbol replacing the division sign were it not for the gain effected by extending the notation in (4). For example,

$$(5) \qquad a : b : c = d : e : f,$$

which is read "a is to b is to c as d is to e is to f," states in briefer form the three proportions: $\frac{a}{b} = \frac{d}{e}$, $\frac{b}{c} = \frac{e}{f}$, and $\frac{a}{c} = \frac{d}{f}$. Evidently the notation of (5) can be extended indefinitely with as many letters as desired on each side of the equality sign.

Since the equation,

$$(6) \qquad ad = bc,$$

obtained by clearing fractions in (3), is linear in each of the letters involved, the solution for any one of them is simple. Evidently $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$, and $c = \frac{ad}{b}$.

* Note that the indicated units merely tell what the numbers represent, and that all algebraic operations are upon *numbers* (as represented by digits or letters).

EXERCISE 57

Find in each case the ratio of the first to the second quantity.

1. 3 feet, 2 yards.

2. 7 feet, 3 miles.

3. 2 miles, 50 yards.

4. 6 ounces, 4 pounds.

5. 1 ton, 100 ounces.

6. 4 hours, 13 seconds.

7. $x, \frac{1}{x}\cdot$

8. $x - 2, 2 - x.$

9. $a^2 + 1, a + \frac{1}{a}\cdot$

Solve for the unknowns.

10. $x : (x - 1) = (x - 1) : (x - 3).$

11. $x : (x + 1) = (x + 2) : (x + 4).$

12. $(x + 2) : (x + 5) = (3x - 1) : (2x + 2).$

13. $(3x + 1) : (2x - 3) = (x + 1) : (2 - 4x).$

14. $(-3x) : (6x + 1) = (4x + 2) : (12x + 9).$

15. $(2 - 3x) : (4x) = (x + 7) : (3x + 1).$

From the following proportions get pairs of simultaneous equations, and solve for the unknowns.

16. $1 : x : 3 = x : 4 : 5y.$

17. $(x + y) : (2x + 3y) : 1 = 1 : 2 : 3.$

18. $(x + y + 1) : (x - y - 2) : (2x + 3y - 1) = 1 : 2 : 3.$

19. $(2x - y + 1) : (x + y - 2) : (3x - y + 1) = 1 : 2 : 3.$

20. $(x + 1) : (2x + 3) : (3x + 5) = 2y : (3y + 2) : (5y + 3).$

21. $(x + y) : (y + z + 1) : (x + y + 1) : z = 1 : 2 : 3 : 4.$

22. $(x + 1) : (y + 2) : (z + 3) : (x + y + z) = 1 : 2 : 3 : 4.$

12.3. *Consequences of a proportion*

Some results following from

$$\frac{a}{b} = \frac{c}{d} \tag{1}$$

are given in equations (2) to (7) below.

When the members of (1) are multiplied by bd we learn that

$$ad = bc, \tag{2}$$

or, *the product of the means equals the product of the extremes.*

When the members of (2) are divided by dc we find that

$$\frac{a}{c} = \frac{b}{d}, \tag{3}$$

or, referring to (1), *the ratio of the numerators equals the ratio of the corresponding denominators.*

If the divisor is ac, we get

$$\frac{b}{a} = \frac{d}{c}, \tag{4}$$

or, *the reciprocals of equal ratios are equal.*

Again, adding 1 to both members of (1) and writing each new member as a fraction, we have

$$\frac{a+b}{b} = \frac{c+d}{d}. \tag{5}$$

Similarly,

$$\frac{a-b}{b} = \frac{c-d}{d}. \tag{6}$$

Dividing corresponding members of (5) and (6), (as justified by the axiom: "When equals are divided by equals, the quotients are equal"), we have

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \tag{7}$$

The student may find further consequences of (1) by getting equations from (3) and (4) corresponding with (5), (6), and (7) as derived from (1), by inverting all members, and by combining members in various ways. Evidently an unending series of equations of an endless degree of complexity stems from the simple equation (1).

Results such as (2) to (7) were formerly given more prominence in algebra texts than at present. Probably one reason

for the decrease in emphasis is the fact that such results are merely what one gets by applying to (1) the guiding principle for work with equations, namely: "Always do to the right side what is done to the left side." Nevertheless, the processes here suggested may lead to results which are interesting, useful, and far from self-evident.

HINT. If, in launching out for himself, the interested student should arrive at a complicated result of (1) such as, say,

$$\frac{a^2 + ab + ac + bc}{a^2 - ab - ac + bc} = \frac{d^2 + bd + cd + bc}{d^2 - bd - cd + bc}, \tag{8}$$

he should not forget the value of frequent arithmetic tests to guard against errors. For instance, when $a = 1$, $b = 2$, $c = 3$, and $d = 6$, (1) is true and so is (8).

EXERCISE 58

(The numbers in this exercise refer to proportions in Art. 12.3.)

1. Noting that (5), (6), and (7) are consequences of (1), get three similar consequences of (3).

2. Get consequences of (4) similar to those of problem 1.

3. By inverting the members of (5), (6), and (7), as well as of the proportions obtained in problems 1 and 2, get nine more consequences of (1).

Test the following proportions with numerical values which satisfy (1), *such as* $a = 1$, $b = 2$, $c = 3$, $d = 6$, or $a = 2$, $b = 3$, $c = 4$, $d = 6$. *If the tests indicate that a given proportion may be a true consequence of* (1), *try to prove that this is so.*

4. $\dfrac{a+b}{a} = \dfrac{c+d}{c}$.

5. $\dfrac{a+b}{a} = \dfrac{c+d}{d}$.

6. $\dfrac{a-b}{a} = \dfrac{c-d}{c}$.

7. $\dfrac{2a-b}{a} = \dfrac{2c-d}{c}$.

8. $\dfrac{a^2}{a+b} = \dfrac{c^2}{c+d}$.

9. $\dfrac{a+2b}{b} = \dfrac{c+2d}{d}$.

10. $\dfrac{(a+b)^2}{ab} = \dfrac{(c+d)^2}{cd}$.

11. $\dfrac{a^2+b^2}{ab} = \dfrac{c^2+d^2}{cd}$.

12. $\dfrac{a^2 - b^2}{ab} = \dfrac{c^2 - d^2}{cd}$.

13. $\dfrac{a^2 - b^2}{a^2} = \dfrac{c^2 - d^2}{c^2}$.

14. $\dfrac{a^3 + b^3}{ab} = \dfrac{c^3 + d^3}{cd}$.

15. $\dfrac{(a + b)^3}{a^2b} = \dfrac{(c + d)^3}{c^2d}$.

Solve the following problems by means of proportions.

16. A light is on the top of a 20-foot pole. If a 6-foot man casts an 8-foot shadow, how far is he from the pole?

17. In problem 16, if the man is 30 feet from the pole, how long will his shadow be?

18. If a light is on top of a pole and a 6-foot man 30 feet away casts a shadow 10 feet long, how high is the pole?

19. A model, 3 feet high, is made from photographs of a mountain. If two villages known to be 5 miles apart are 7 feet apart on the model, how high is the mountain?

12.4. *Variation*

Two quantities which behave so that their ratio remains fixed may be compared mathematically by use of the symbol $\propto$, read "varies as." Thus, if C and D represent respectively the circumference and diameter of a circle, then C is doubled when D is doubled, tripled when D is tripled, and so on. This is expressed in English by saying that "C varies directly as D," or, "C is directly proportional to D." More simply, we may say that "C varies as D," or "C is proportional to D." In symbols,

$$C \propto D. \tag{1}$$

The statement (1) can be put in a form more suitable for algebraic operations as the equation

$$C = kD, \quad \text{or} \quad \frac{C}{D} = k, \tag{2}$$

where k is called the "constant of proportionality," or the "variation constant." In this case, when C and D are meas-

ured in terms of the same linear unit, it is known that $k = \pi$ or (nearly) 3.1416, so that we have the formula

$$(3) \qquad C = \pi D.$$

In many problems, however, we must start with an undetermined k. Thus at a given time of day in a given place the length, s, of a man's shadow varies as his height, h. We then have

$$(4) \qquad s = kh,$$

where k remains to be found. If the shadow of a 6-foot man is 9 feet long, then

$$(5) \qquad 9 = k6, \quad \text{or} \quad k = \tfrac{9}{6} = \tfrac{3}{2}.$$

Hence, at the time and place in question,

$$(6) \qquad s = \tfrac{3}{2}h$$

for people or objects of various heights. In other words, the shadow will be $\frac{3}{2}$ as long as the height of the object.

Sometimes one quantity varies, not as a second one, but as some function of one or more quantities. Thus, the equations

$$(7) \qquad y = kx^2, \quad \text{or} \quad \frac{y}{x^2} = k,$$

$$(8) \qquad y = kxz, \quad \text{or} \quad \frac{y}{xz} = k,$$

$$(9) \qquad y = \frac{k}{x}, \quad \text{or} \quad xy = k,$$

and

$$(10) \qquad y = \frac{kx}{z^2w}, \quad \text{or} \quad \frac{yz^2w}{x} = k,$$

state in succession that y varies directly as x^2, xz, $\frac{1}{x}$, and $\frac{x}{z^2w}$. Sometimes it is said that, in (8), y varies *jointly* as x and z; and in (10), y varies *directly* as x and *inversely* as z^2 and w. More simply, however, *if y varies directly as any function of other letters, y is k times that function.*

Other common forms of expression for the variation law

connecting y with the other variables in (7), (8), (9), and (10) can be used. In order to acquaint the student with some of these phrases, we shall list a few. In (7), it can be said that "y varies as x^2," or that "y is directly proportional to x^2," or simply that "y is proportional to x^2." All of these statements imply that $\frac{y}{x^2}$ is a constant.

Similarly, in (8), it can be said that "y varies directly as the product of x and z," or that "y is directly proportional to the product of x and z," or, more simply, that "y is proportional to x and z." All of these statements imply that $\frac{y}{xz}$ is a constant.

In (9), it can be said that "y varies inversely as x" or that "y is inversely proportional to x." These statements imply that xy is a constant.

In (10), it can be said that "y varies directly as x and inversely as the product of z^2 and w" or that "y is directly proportional to x and inversely proportional to z^2 and w." These statements imply that $\frac{yz^2w}{x}$ is a constant.

It should be observed that the words "directly" and "jointly" are frequently omitted, "directly" being understood when the dependence of one quantity on another is referred to, and "jointly" when there is more than one other quantity involved.

A problem in variation usually consists of the following steps:

1. The variation relation is stated as an equation involving the constant k. In this step all the quantities involved in the variation law should be described clearly. When any quantity is referred to by name, we shall adopt the plan of using the initial letter of the name as an abbreviation. For example, the word "pressure" would be described by the letter p or P. When the quantities are not named, the letters x, y, z, etc., are used.

2. One set of values of all the variables involved is inserted in the equation, leaving k as the only unknown. It is advisable in a problem where step 5 is to be carried out later, to make a box scheme as indicated in Example 4, solution 3. This box should be filled in before going through any numerical work, as it will clearly show what quantities are given, the units being used respectively, and what quantities are to be found.

3. The resulting equation is solved for k.

4. The value of k is inserted in the original equation.

5. By means of the equation obtained in 4, the value of any specified variable is found when the values of the others are given. However, the same set of units must be used throughout the problem as were employed in steps 2 and 3 to find the value of k. This point is very important, and is illustrated in the two versions of Example 4.

Example 1. y varies directly as $\frac{x^2}{zw}$. One set of values is: $x = 2$, $y = 4$, $z = -3$, and $w = -2$. Find z when $x = 3$, $y = -1$ and $w = 4$.

Solution. The five steps in order yield the following equations.

$$y = \frac{kx^2}{zw}. \tag{11}$$

$$4 = \frac{4k}{(-3)(-2)}. \tag{12}$$

$$k = 6. \tag{13}$$

$$y = \frac{6x^2}{zw}. \tag{14}$$

$$-1 = \frac{6(3)^2}{z(4)}, \quad \text{or} \quad z = \frac{-27}{2} \text{ (answer)}. \tag{15}$$

Example 2. State in words the law of variation for x in terms of the variables related to it in (8) and (10).

Solution. If we use the second forms in (8) and (10), where the left members include all the variables present, it becomes

apparent that in a direct variation between two variables, one appears as a divisor of the numerator while the second appears as a divisor of the denominator. For inverse variation, both variables will appear in the same position either as divisors of the numerator or as divisors of the denominator. Thus, in (8)

"x varies directly as y and inversely as z."

Similarly in (10)

"x varies directly as y, z^2, and w."

It should be noted here that there are other forms of expression as well as those given. They are left to the student.

Example 3. If the original value of x is 10 units in (8) and (10), what would its new value become if y is doubled, z is tripled, and w is halved?

Solution. In (8), since x varies directly as y and inversely as z, it means that if y is doubled, so is x, but if z is tripled, x becomes one-third of its former measure. Carrying out these changes simultaneously, we get for x,

$$2(\tfrac{1}{3})(10) = \tfrac{20}{3} = 6\tfrac{2}{3} \text{ units.}$$

In (10), since x varies directly as y, z^2, and w, it means that if y is doubled, so is x; if w is halved, so is x; but if z is tripled, x becomes nine times its former measure. Carrying out these changes simultaneously, we get for x,

$$2(\tfrac{1}{2})(9)(10) = 90 \text{ units.}$$

Example 4. Boyle's Law in physics states that at a given temperature the volume of a given quantity of gas is inversely proportional to the pressure on the walls of the container. If for a given sample the volume is one cubic foot when the pressure is 20 pounds per square inch, what is the volume when the pressure is increased to 60 pounds per square inch?

Solution 1. Let

$$V = \frac{k}{P} \quad \text{or} \quad VP = k, \tag{16}$$

where V and P represent units of volume and pressure respectively. In this first solution let P be measured in pounds per square inch, and V in cubic feet. Then

$$1 = \frac{k}{20}. \tag{17}$$

$$k = 20. \tag{18}$$

$$V = \frac{20}{P}. \tag{19}$$

$$V = \frac{20}{60} = \frac{1}{3}. \tag{20}$$

Solution 2. Now let P be measured in pounds per square foot, with V unchanged in meaning. Then the first and second values of P in the problem are respectively $\frac{20}{144} = \frac{5}{36}$, and $\frac{60}{144} = \frac{5}{12}$. Equations (17) to (20) are replaced by the following.

$$1 = \frac{k}{\frac{5}{36}}. \tag{21}$$

$$k = \frac{5}{36}. \tag{22}$$

$$V = \frac{5}{36P}. \tag{23}$$

$$P = \frac{5}{36(\frac{5}{12})} = \frac{1}{3}, \text{ as before.} \tag{24}$$

Thus, we see that the value determined for k, the so-called "proportionality constant," depends upon the choice of units; but *the same final result* is obtained in any case.

Solution 3. Simple problems in variation may often be worked more easily as problems in proportion; for if one quantity varies as another, corresponding values are proportional. When using the proportion method, it is better to write the variation equation in the second form illustrated in (7), (8), (9), and (10), where all variables are contained

in one member of the equation. Thus, using the second form of (16) with V_1, P_1 indicating one set of values for V and P, while V_2, and P_2 a second set, etc., it follows that $V_1P_1 = V_2P_2 = V_3P_3$, etc. Arrange the given information in a box scheme as follows:

V (cu. ft.)	P (lbs. per sq. in.)
$V_1 = 1$	$P_1 = 20$
$V_2 = ?$	$P_2 = 60$

LAW:
$$V_1P_1 = V_2P_2 = V_3P_3 = \text{etc.}$$
$$1 \cdot 20 = V_2(60).$$
$$\therefore\ V_2 = \tfrac{1}{3} \text{ cu. foot.}$$

Example 5. Here we shall use without proof the following important and useful mathematical result:

THEOREM 1. *The areas of similar plane figures vary as the squares of corresponding dimensions.*

The areas of two similar rectangles are 15 and 20 square units, respectively. Find the length of the diagonal of the larger rectangle if the diagonal of the smaller one is 10 units long.

Solution. Let A and d stand for area and diagonal length respectively. The mathematical statement of the theorem here used is

$$(25) \qquad A = kd^2, \quad \text{or} \quad \frac{A}{d^2} = k.$$

From the smaller rectangle we have

$$(26) \qquad 15 = k(10)^2, \quad \text{or} \quad k = \frac{15}{100} = \frac{3}{20}.$$

This value, substituted in (25), gives

$$(27) \qquad A = \frac{3d^2}{20}.$$

Inserting the area of the larger rectangle, or $A = 20$, in (27), and remembering that in this case d must be positive, we have

$$d = 20\frac{\sqrt{3}}{3} = 11.5 \text{ linear units (answer).} \tag{28}$$

Note. If the student will try to solve this problem by means of simultaneous equations, as is possible, he will appreciate the simplicity and power of this method. In addition to the required answer he has in (27) a formula applying to *any* rectangle similar to the first one.

In Exercise 59 we shall use, in addition to Theorem 1,

THEOREM 2. *The volumes of similar figures vary as the cubes of corresponding dimensions.*

EXERCISE 59

In each of the problems 1–8,

(*a*) *express the given relation as an equation;*
(*b*) *write the equation so that the left member will contain all the variables; and then*
(*c*) *write an equivalent statement in words for the variation involving x as related to the other variables present.*

1. $y \propto x^3$. **2.** $z \propto xy$. **3.** $w \propto \dfrac{x^2y}{z}$. **4.** $y \propto \dfrac{w}{x^2z}$.

5. $z \propto \dfrac{1}{xy}$. **6.** $w \propto \dfrac{x}{yz^3}$. **7.** $x^2 \propto \dfrac{y}{z^2w}$. **8.** $y^3 \propto \dfrac{xz}{w}$.

9. If z varies as $\dfrac{x^2y}{w}$ and is 4 when $x = -2$, $y = 3$, and $w = -4$, find the value of

(a) y when $x = 3$, $w = -2$, and $z = 4$;
(b) x when $z = 5$, $y = 10$, and $w = 8$;
(c) w when $x = -3$, $y = 4$, and $z = -5$;
(d) z when $x = -3$, $y = 4$, and $w = -2$.

10. If y varies as $\dfrac{z}{wx^2}$ and is -2 when $z = 3$, $w = 2$, and $x = -4$, find the value of

(a) y when $z = -3$, $w = -4a$, and $x = -b$;
(b) z when $y = -5$, $x = 2a$, and $w = -3c$;
(c) w when $y = -2$, $z = -3$, and $x = 2a$;
(d) x when $y = a$, $z = b$, and $w = c$.

Solve problems 11–15 *by use of Theorems* 1 *and* 2, *Art.* 12.4.

11. The bases of 6 similar triangles are 2, 3, 4, 7, 8, and 9 inches long respectively. If the area of the smallest triangle is 5 square inches, what are the other areas?

12. A sign painter finds that a pint of paint is used in painting the words, "Chicken Dinner. Dine and Dance." How much paint will be used in making a similar sign, with letters of the same type and proportions but three times as high?

13. The heights of three similarly-proportioned men are 5, $5\frac{1}{2}$, and 6 feet respectively. If the tallest man weighs 200 pounds, about what would be the expected weights of the other two?

14. The weights of various aerial bombs in pounds are as follows: 100, 200, 500, 1000, 2000, 6000, and 12,000. Assuming that all are similarly proportioned and made of the same materials in like proportions, what are the lengths of the other bombs if the 100-pound one is 2 feet long?

15. The perimeters of 6 similar plane figures are 4, 5, 10, 20, 50, and 100 inches, respectively. If the area of the figure with a 20-inch perimeter is 10 square inches, what are the other areas?

16. The weight of any given object on a planet varies directly as the mass of the planet and inversely as the square of its radius. For reference let the mass of the earth be one unit and its radius one unit. If a boy weighs 100 pounds on the earth, how much would he weigh on each of the following bodies?

Name	*Radius*	*Mass*
Moon	.27	.012
Mercury	.39	.04
Venus	.97	.81
Mars	.53	.11
Jupiter	11.2	317.
Saturn	9.4	95.
Sun	109.	330,000

17. The weight of a body within the earth, as in a mine, varies directly as its distance from the center. Assuming the radius of the earth to be 4000 miles, how much would a 200-pound man weigh when 10 miles below the surface?

18. If S varies directly as the cube of x and inversely as the square of y, what change in S results when x is doubled and y is tripled?

19. The force of attraction between two bodies varies directly as the product of their masses and inversely as the square of the distance between them. When two masses of 6 units and 24 units are separated by 192 inches, the force is 72 units. (a) If the distance between them is diminished by 24 inches and the smaller mass is doubled, what change in the larger mass will increase the force by 18 units? (b) At what distance will the two given masses have twice the force of attraction that they have at 192 inches?

20. Kepler's Law states that the square of the time of a planet's revolution about the sun varies as the cube of its mean distance from the sun. The mean distances of Mars and the earth are in the ratio 3 : 2. Find in days the time of revolution of Mars (i.e., the Martian year).

21. The volume V of a gas varies directly as the absolute temperature T and inversely as the pressure P. If a certain amount of gas occupies 100 cubic feet at a pressure of 16 pounds per square inch and at $T = 200°$, find its volume when the pressure is 20 pounds per square inch and $T = 420°$.

22. The mass of a spherical body varies directly as its density and the cube of the radius. Compare the masses of Jupiter and the earth if the diameter of Jupiter be taken as 11 times that of the earth and its density $\frac{5}{22}$ that of the earth.

23. Given that $y \propto x$, prove that $2x^2 + y^2 \propto xy$.

HINT. Let $y = kx$.

Show that $\dfrac{2x^2 + y^2}{xy}$ = a constant.

24. If $y \propto x$, prove that $3x^4 + y^2x^2 \propto 3x^2y^2$.

Chapter Thirteen

THE BINOMIAL THEOREM

13.1. *The binomial theorem*

If we carry through the multiplications indicated in the left side of the expressions below, we get the following formulas.

$$(1)\quad \begin{cases} (F+S)^2 = F^2 + 2FS + S^2; \\ (F+S)^3 = F^3 + 3F^2S + 3FS^2 + S^3; \\ (F+S)^4 = F^4 + 4F^3S + 6F^2S^2 + 4FS^3 + S^4; \\ (F+S)^n = F^n + nF^{n-1}S + \dfrac{n(n-1)}{1\cdot 2}F^{n-2}S^2 \\ \quad + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}F^{n-3}S^3 + \cdots \text{ (to } n+1 \text{ terms).} \end{cases}$$

The right member in each of equations (1) is said to be the binomial expansion of its corresponding left member. The expansion of $(F + S)^n$ is called the *binomial formula;* and the full equation is the algebraic statement of *the binomial theorem dealing with a positive integral index, n.* The proof of the binomial formula is given in more advanced discussions, but the student may verify it for various integral values of n.

It will be noted that if n represents the exponent of $F + S$ in the left member of any of the equations of (1), then the following statements are true:

1. The number of terms in the expansion is $n + 1$.
2. The first term is F^n.
3. The exponent of F decreases by one in each succeeding term of the expansion, while that of S increases by one.

Furthermore, the sum of the exponents of F and S in any term is n.

4. The coefficient of the first term is 1, while the coefficient of the second term is n. If we multiply the coefficient of any term by the exponent of F in that term, and then divide the product by one more than the exponent of S in the same term, the result is the coefficient of the next term in the expansion.

5. The coefficients equidistant from both ends of the expansion are equal.

Example 1. Expand $(x + y)^7$.

Solution.

$$(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$$

The first coefficient is 1.

Since $n = 7$ in this case, the second coefficient is 7.

The coefficient of the third term is $\frac{(7)(6)}{2}$ or 21.

The coefficients of the fourth to eighth terms, as found in succession, are:

$$\frac{(21)(5)}{3} = 35;\ \frac{35(4)}{4} = 35;\ \frac{(35)(3)}{5} = 21;$$
$$\frac{(21)(2)}{6} = 7;\text{ and } \frac{(7)(1)}{7} = 1.$$

Example 2. Expand and simplify $(x - 2y)^5$.

Solution. Comparing the binomial $x - 2y$ with $F + S$, we find that x corresponds with F and $-2y$ with S. By the binomial formula,

$$\begin{aligned}[x+(-2y)]^5 &= x^5+5(x)^4(-2y)+10x^3(-2y)^2+10x^2(-2y)^3 \\ &\quad +5x(-2y)^4+(-2y)^5 \\ &= x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5.\end{aligned}$$

Example 3. Expand and simplify $\left(3\sqrt{x} + \frac{y}{2}\right)^4$.

Solution. Here $F = 3\sqrt{x}$ and $S = \frac{y}{2}$. Again using the binomial formula, we have

$$\begin{aligned}\left(3\sqrt{x} + \frac{y}{2}\right)^4 &= (3\sqrt{x})^4 + 4(3\sqrt{x})^3\left(\frac{y}{2}\right) + 6(3\sqrt{x})^2\left(\frac{y}{2}\right)^2 \\ &\quad + 4(3\sqrt{x})\left(\frac{y}{2}\right)^3 + \left(\frac{y}{2}\right)^4 \\ &= 81x^2 + 54x^{\frac{3}{2}}y + \frac{27xy^2}{2} + \frac{3x^{\frac{1}{2}}y^3}{2} + \frac{y^4}{16}.\end{aligned}$$

13.2. *An extension of the binomial formula*

It is interesting to note that if n is either negative or fractional in $(F + S)^n$, the binomial formula still holds when proper restrictions have been placed on the numerical values of F and S. Further, the expansion is not limited to $n + 1$ terms as before, but is made up of an infinite number. It is beyond the scope of this text to discuss these restrictions on the binomial expansion when n is not a positive integer. However, it may be stated in general that the expansion is useful when S is numerically smaller than F, and when the successive terms approach zero so rapidly that the sum of the first three or four terms of the expansion serves as a good approximation for $(F + S)^n$. Example 3 below shows an application to the extraction of roots.

Example 1. Expand $(x + y)^{-2}$ to five terms and simplify.

Solution.

$$\begin{aligned}(x + y)^{-2} &= x^{-2} + (-2)x^{-3}y + \frac{(-2)(-3)}{2}x^{-4}y^2 \\ &\quad + \frac{(-2)(-3)(-4)}{2 \cdot 3}x^{-5}y^3 \\ &\quad + \frac{(-2)(-3)(-4)(-5)}{2 \cdot 3 \cdot 4}x^{-6}y^4 + \cdots \\ &= x^{-2} - 2x^{-3}y + 3x^{-4}y^2 - 4x^{-5}y^3 + 5x^{-6}y^4 - \cdots\end{aligned}$$

Example 2. Expand $(x - y)^{-\frac{1}{2}}$ to four terms and simplify.

Solution.

$$(x - y)^{-\frac{1}{2}} = x^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}(-y) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}x^{-\frac{5}{2}}(-y)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{2 \cdot 3}x^{-\frac{7}{2}}(-y)^3 + \cdots$$

$$= x^{-\frac{1}{2}} + \frac{x^{-\frac{3}{2}}y}{2} + \frac{3x^{-\frac{5}{2}}y^2}{8} + \frac{5x^{-\frac{7}{2}}y^3}{16} + \cdots$$

Example 3. Find $\sqrt[3]{1002}$ to three decimal places.

Solution. Write

$$\begin{aligned}\sqrt[3]{1002} &= (1000 + 2)^{\frac{1}{3}} = [1000(1 + .002)]^{\frac{1}{3}} \\ &= 10(1 + .002)^{\frac{1}{3}} \\ &= 10\left[1^{\frac{1}{3}} + \frac{1}{3}(1)^{-\frac{2}{3}}(.002) + \frac{\frac{1}{3}(-\frac{2}{3})(1)^{-\frac{5}{3}}(.002)^2}{2} + \cdots\right] \\ &= 10(1 + .00067 - .0000004 + \cdots) \\ &= 10.0000 + .0067 - .0000 + \cdots \\ &= 10.0067 \text{ or } 10.007.\end{aligned}$$

Note that for accuracy to three decimal places we write successive terms to four decimal places until a term is reached which yields zeros in the four places. Then the answer is "rounded off" to three places. This is a good working rule, though a much more extended discussion would be needed for a rigorous treatment of problems of this type.

EXERCISE 60

Expand and simplify each of the following expressions.

1. $(x + y)^3$.
2. $(x - y)^3$.
3. $(2x - y)^6$.
4. $(3a^2 + b)^3$.
5. $\left(2x - \frac{3}{y}\right)^3$.
6. $(2\sqrt{x} + y)^4$.
7. $\left(3x - \frac{y}{2}\right)^6$.
8. $\left(a^2 + \frac{b}{3}\right)^4$.
9. $(2x - y^2)^5$.
10. $(3\sqrt{x} - y)^5$.
11. $\left(\frac{x}{2} + \frac{y^2}{3}\right)^4$.
12. $\left(\frac{x^{\frac{1}{3}}}{2} - y\right)^3$.

Find the first four terms in each of the following indicated expansions, and simplify the terms obtained.

13. $(x^{\frac{1}{2}} - 2y)^{10}$. **14.** $\left(\frac{2}{x} - \frac{x}{4}\right)^8$. **15.** $\left(\sqrt{x} + \frac{\sqrt[3]{y}}{2}\right)^{12}$.

Find and simplify the required terms in the following indicated expansions.

16. 5th term of $\left(\frac{x}{y} - \frac{y}{x}\right)^8$.

HINT. Let t_5 represent the fifth term, with $\frac{x}{y}$ corresponding to F and $\frac{-y}{x}$ to S.

Then $t_5 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{x}{y}\right)^4\left(\frac{-y}{x}\right)^4$.

17. 6th term of $\left(ab - \frac{b^2}{a}\right)^{12}$.

18. 8th term of $(\sqrt{x} - x^{-\frac{1}{2}})^{14}$.

Assuming that all restrictions have been taken into account, expand to four terms and simplify.

19. $(x + y)^{-3}$. **20.** $(x - y)^{\frac{1}{2}}$. **21.** $(2x + y)^{-\frac{1}{2}}$.

22. $(a^{\frac{1}{2}} + 2b^{\frac{1}{2}})^{-1}$. **23.** $(1 + 3x)^{-\frac{2}{3}}$. **24.** $(1 - 2x)^{-1}$.

By grouping terms, express as binomials and expand by use of the binomial formula.

25. $(a + b + c)^2$, or $[(a + b) + c]^2$. **26.** $(a + b - c - d)^2$.

27. $(a + b - c)^2$. **28.** $(a - b - c)^2$.

29. $(a + b + c)^3$. **30.** $(a + b - c)^3$.

Use the method of illustrative Example 3, *Art.* 13.2, *to find the values of the following roots to three decimal places.*

31. $\sqrt{101}$. **32.** $\sqrt{110}$. **33.** $\sqrt[3]{1.005}$. **34.** $\sqrt[3]{1010}$.

35. $\sqrt{26}$. **36.** $\sqrt{24}$ (or $\sqrt{25 - 1}$). **37.** $\sqrt[3]{10}$ (or $\sqrt[3]{8 + 2}$).

★

Chapter Fourteen

PROGRESSIONS

14.1. *Definitions*

A group of numbers arranged in order according to some law of selection is called a *sequence.* Each number in the sequence is a *term.*

14.2. *Arithmetic progressions*

If the difference between two adjacent terms is the same, regardless of the pair chosen, the sequence is called an *arithmetic progression.*

Example. 1, 3, 5, 7, 9, 11.

The abbreviation "A.P." stands for an *arithmetic progression.*

The letters a, d, n, l, and s are used to represent the *elements* of an A.P., where

a represents the first term;

d represents the *common difference,* or what must be added to any term to get the next one;

n represents the number of terms;

l represents the last, or nth, term; and

S represents the sum of all the n terms.

It may aid the memory to note that the elements are the letters in the word "lands."

These elements are so related that if any three of them are known the remaining two can be found by use of the proper formulas.*

* In some cases, when only the three formulas for an A.P. given in the text are used, it is necessary to solve simultaneously two equations with two unknowns. We shall omit examples of this type.

Consider first the progression.

(1) $a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, a + 6d.$

Here any term after the first is found by adding d to the term which precedes it. Note that second term is $a + 1d$, the third term is $a + 2d$, etc., so that the coefficient of d in any term is one less than the ordinal number of the term. From this we see that the nth term, designated by l, is $a + (n - 1)d$. Stated in symbols, this yields an important result which we shall call Formula 1, namely,

1. $\boldsymbol{l = a + (n - 1)d.}$

A progression having a definite number of terms may be written out in full, but usually some dots are inserted to indicate that some of the terms are omitted.

Example 1. 2, 5, 8, $\cdots$ to 8 terms.

Example 2. 1, 3, 5, $\cdots$ 19, 21.

14.3. *Arithmetic means and extremes*

The first and last terms of a progression are called the *extremes*, and all terms between the extremes are called *means*.

In the special case of an A.P. having only three terms, the middle term is called *the* arithmetic mean.

Example. Find the arithmetic mean of a and l.

Solution. Let m be the desired mean in the A.P.: a, m, l. Since $m - a = d$, and also $l - m = d$, it follows that

(1) $$m - a = l - m.$$

Solving for m we have, as Formula 2,

2. $$\boldsymbol{m = \frac{a + l}{2}}.$$

In words, *the arithmetic mean of two numbers is their arithmetic average.*

If there is more than one mean to be inserted between two given extremes, we substitute the known values of a, l, and n in Formula 1 and solve for d. The entire set can then be written by adding d to each successive term, beginning with the first, until the last term (the second extreme) is reached.

Example 1. Find the mean of 2 and 8.

Solution. By Formula 2 (here, and in the other examples to follow, the student should *first of all* write out the formula in question),

$$m = \frac{2 + 8}{2} = \frac{10}{2} = 5. \tag{2}$$

Example 2. Find the mean of 7 and -15.

Solution. $m = \dfrac{7 + (-15)}{2} = \dfrac{-8}{2} = -4.$

Example 3. Insert three means between 3 and 11.

Solution. Here $a = 3$, $l = 11$, and $n = 5$. By Formula 1,

$$11 = 3 + 4d, \tag{3}$$

whence

$$d = 2. \tag{4}$$

The progression, then, as obtained by adding 2 to each successive term, is 3, 5, 7, 9, 11; and the required means are 5, 7, and 9 (answer).

Example 4. Insert four means between 1 and -14.

Solution. Here $a = 1$, $l = -14$, and $n = 6$. By Formula 1,

$$-14 = 1 + 5d, \tag{5}$$

whence

$$d = -3. \tag{6}$$

The progression is 1, -2, -5, -8, -11, -14; and the means are -2, -5, -8, and -11 (answer).

Example 5. Find the 7th term of the progression: 2, 5, 8, $\cdots$.

Solution. From the given terms we see that $a = 2$ and $d = 3$. Also $n = 7$. By Formula 1,

$$l = 2 + (6)(3) = 20 \text{ (answer).} \tag{7}$$

Example 6. The first 3 terms of an A.P. having 8 terms are 2, -1, and -4. Find the last term.

Solution. Here $a = 2$, $d = -3$, and $n = 8$. By Formula 1,

$$l = 2 + 7(-3) = -19 \text{ (answer).} \tag{8}$$

EXERCISE 61

Find the arithmetic mean of the numbers given.

1. 3 and 19.

2. 2 and -12.

3. $3x$ and $8x$.

4. -3 and -9.

5. $5y^2$ and $4y^2$.

6. Insert 2 means between 3 and 15.

7. Insert 3 means between 2 and -22.

8. Insert 5 means between 10 and -2.

9. Find the 10th term of the progression: 3, 5, 7 $\cdots$.

10. If $l = 21$, $n = 7$, and $d = 3$, find a.

11. If $n = 8$, $l = 30$, and $a = 2$, find d.

12. There are 5 apple trees in a row. Each tree, after the first, produces 10 more apples than the one that precedes it. If the first tree produces 75 apples, how many are obtained from the last tree?

13. The first term of an A.P. is 21. If $d = -3$ and $l = 0$, how many terms are there?

14. Find the 10th term of the progression: 3, 5, 7, $\cdots$.

15. A group of boys, counting their marbles, found that one had 27, the next had 24, and so on to the last boy, who had 12. How many boys were there?

16. A man travels 50 miles on Jan. 1, 55 miles Jan. 2, and so on through the month. How far did he travel on the last day of January?

17. A man traveled 100 miles on Monday, 85 miles Tuesday, and so on through the week. How far did he travel on Saturday?

18. A man invested \$10 more each month, after the first one, than he invested in the preceding month. His investment for the 10th month was \$110. What was the first month's investment?

14.4. *The sum of an A.P.*

We could find the sum of the terms of an A.P. by adding them; but that would be a long process in many cases. It is shorter and more convenient to use the formula for the sum derived from the general expressions for S in the two equations below. The second one is obtained by reversing the order of the terms in the first one.

(1) $S = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l.$

(2) $S = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$

Adding corresponding members of (1) and (2), we have

(3)
$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l).$$

Noting that $a + l$ occurs n times in this sum, we may write

(4) $$2S = n(a + l),$$

and solving (4) for S we have Formula 3:

3. $\boldsymbol{S = \frac{n}{2}(a + l).}$

A second formula for the sum may be obtained by replacing the l in Formula 3 by $a + (n - 1)d$, its value by Formula 1. Thus, $S = \frac{n}{2}[a + a + (n - 1)d]$ so that Formula 4 is:

4. $\boldsymbol{S = \frac{n}{2}[2a + (n - 1)d].}$

To find S we use Formula 3, of course, when n, a, and l are given, and Formula 4 when we know n, a, and d.

Example 1. Find the sum of the first 6 terms of the progression: 1, 3, 5, $\cdots$.

Solution. Here $a = 1$, $d = 2$, and $n = 6$. By Formula 4,

$$(5) \quad S = \tfrac{6}{2}[2(1) + (6 - 1)2] = 3(2 + 10) = 36 \text{ (answer)}.$$

Example 2. Find the sum of the first 5 terms of an A.P. in which $a = 2$ and $l = 14$.

Solution. By Formula 3, with $a = 2$, $l = 14$, and $n = 5$,

$$(6) \qquad S = \tfrac{5}{2}[(2 + 14)] = \tfrac{5}{2}(16) = 40 \text{ (answer)}.$$

EXERCISE 62

1. Find the sum of the first 6 terms of the A.P.: 3, 5, 7, $\cdots$.

2. Find the sum of the first 10 terms of the A.P.: 32, 29, 26, $\cdots$.

3. Find the sum of the first 8 terms of the A.P.: -25, -21, -17, $\cdots$.

4. Find the sum of the first 12 terms of the A.P.: -7, -4, -1, $\cdots$.

5. The first term of an A.P. is 5 and the 7th term is 17. Find the sum of the first 7 terms.

6. If $a = 4$, $l = 23$, and $s = 243$, find n and d.

7. If $a = 5$, $n = 12$, and $d = 2$, find S and l.

8. If $a = 12$, $d = 3$, and $l = 30$, find S and n.

9. If $a = 101$, $S = 1457$, and $l = -7$, find n and d.

10. If $S = 24$, $d = -2$, and $l = -4$, find a and n.

11. If $n = 13$, $d = \frac{3}{2}$, and $l = 33$, find S and a.

12. If $a = 17$, $n = 15$, and $l = -25$, find S and d.

13. If $S = 171$, $n = 9$, and $d = 2$, find a and l.

14. If $a = 5$, $d = 3$, and $S = 98$, find n and l.

15. If $n = 10$, $l = 43$, and $S = 250$, find a and d.

16. If $a = -6$, $n = 11$, and $S = 99$, find d and l.

17. How many boys are required for a triangular formation having 6 boys in the first row, 5 in the next, and so on to the last row, in which there is only one boy?

18. A ball falls and bounces to a height of 25 feet. It continues bouncing 5 feet less each time. How far does it travel between the first and sixth time it strikes the ground?

19. Ten bales of hay are lying 4 feet apart in a row. The first is 4 feet from a truck, the second is 8 feet, etc. How far must a man travel if he starts at the truck and carries it all, one bale at a time, to the truck?

20. A boy starts at the first one of 11 marks that are 5 yards apart and touches each of the other ten marks in order. If he returns to his starting point after each touch, what is the total distance he travels?

21. Five boys inherit an estate. Each one after the first receives \$200 less than the one before him. If the value of the estate was \$13,000 how much did each boy receive?

22. A man invests \$100 at 5% simple interest at the beginning of each year. Find the value of his investments at the end of 10 years.

23. A man invests \$1000 at 4% simple interest at the beginning of each year. Find the value of his investments at the end of 6 years.

24. A man invests \$500 at 6% simple interest at the beginning of each year. Find the value of his investments at the end of 5 years.

14.5. *Geometric progressions*

The following sequence is called a *geometric progression*, or as abbreviated, a G.P.

$$a, ar, ar^2, ar^3, \cdots, l. \tag{1}$$

From (1) we note that any term after the first is found by multiplying the term preceding it by r. Or again, if any term after the first is divided by the preceding term, the quotient is r. This quantity, r, is called the *common ratio*.

Aside from r, which replaces the d of an A.P. the elements of a G.P. are the same as those of an A.P. The word "snarl" (replacing "lands"), here might be used as a memory aid.

The exponent of r in any term is one less than the ordinal number of the term. It follows that if there are n terms in a G.P., the last term is ar^{n-1}. Thus, we have Formula 5:

5. $\boldsymbol{l = ar^{n-1}}$.

The sum S of the terms of a G.P., called more briefly the sum of the G.P., may be indicated as follows:

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \tag{2}$$

If we multiply each member of (2) by r, we have

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{3}$$

When the left and right sides of (3) are subtracted from the corresponding sides of (2), all terms in the right members drop out except a in (2) and ar^n in (3). Thus, we get

$$S - rS = a - ar^n, \tag{4}$$

or

$$S(1 - r) = a(1 - r^n). \tag{5}$$

The solution of (5) for S yields Formula 6:

6. $\boldsymbol{S = \dfrac{a(1 - r^n)}{1 - r} \qquad (r \neq 1).}$

Note that if $r = 1$, Formula 6 gives S the meaningless value $\frac{0}{0}$, whereas actually $S = na$ in that case.

Since $l = ar^{n-1}$ by Formula 5, it follows that $rl = ar^n$. Replacing ar^n by rl in Formula 6, we get Formula 7:

7. $\boldsymbol{S = \dfrac{a - rl}{1 - r} \qquad (r \neq 1).}$

If any three of the elements a, r, n, l, and S are given, the other two can be found by use of Formulas 5, 6, and 7.

Example 1. Find the 10th term of the progression: 1, 2, 4, $\cdots$.

Solution. Here $a = 1$, $r = 2$, and $n = 10$. By Formula 5,

$$l = 1(2)^9 = 512 \text{ (answer)}. \tag{6}$$

Example 2. Find the sum of the first 8 terms of the progression: 2, 6, 18, $\cdots$.

Solution. By Formula 6, with $a = 2$, $r = 3$, and $n = 8$,

$$S = \frac{2(1 - 3^8)}{1 - 3} = \frac{2(3^8 - 1)}{3 - 1} = 3^8 - 1 = 6560 \text{ (answer)}. \tag{7}$$

Example 3. Insert 3 geometric means between $\frac{1}{2}$ and 8.

Solution. Here $a = \frac{1}{2}$, $l = 8$, and $n = 5$. By Formula 5,

$$8 = \tfrac{1}{2}(r)^4, \tag{8}$$

or

$$16 = r^4. \tag{9}$$

Since 2 and -2 are fourth roots of 16, we have $r = \pm 2$. The G.P. is then either $\frac{1}{2}$, 1, 2, 4, 8 or $\frac{1}{2}$, -1, 2, -4, 8, and hence the geometric means are 1, 2, and 4 or -1, 2, and -4.

(It should be noted that in this chapter we are disregarding any solutions in which the value of r involves the imaginary unit i.)

Example 4. Find the geometric mean of x and y, assuming that both are positive or both are negative.

Solution. Here $a = x$, $l = y$, and $n = 3$. By Formula 5,

$$y = x(r)^2, \tag{10}$$

whence

$$r^2 = \frac{y}{x}, \tag{11}$$

or

$$r = \pm\sqrt{\frac{y}{x}} = \pm\frac{\sqrt{xy}}{x}. \tag{12}$$

The G.P., then, is either $x, \sqrt{xy}, y$ or $x, -\sqrt{xy}, y$. Thus, we find that there are actually two geometric means of any two numbers with like signs. If we denote both of these by M_g, we have, as Formula 8,

8. $\boldsymbol{M_g = \pm\sqrt{xy}}$.

Example 5. Find the geometric means of 4 and 16.

Solution. By Formula 8,

$$M_g = \pm\sqrt{(4)(16)} = \pm\sqrt{64}. \tag{13}$$

Thus, the answers are 8 and -8.

Example 6. If $a = 32$, $l = 1$, and $r = \frac{1}{2}$, find n and S.

Solution. By Formula 5,

$$1 = (32)(\tfrac{1}{2})^{n-1}, \tag{14}$$

whence

$$\tfrac{1}{32} = (\tfrac{1}{2})^{n-1}. \tag{15}$$

But $(\frac{1}{2})^5 = \frac{1}{32}$, so that $n - 1 = 5$, and hence

$$n = 6 \text{ (answer)}. \tag{16}$$

Also, by Formula 7,

$$S = \frac{32 - \frac{1}{2}(1)}{1 - \frac{1}{2}} = \frac{32 - \frac{1}{2}}{\frac{1}{2}} = \frac{64 - 1}{1} = 63 \text{ (answer)}. \tag{17}$$

Example 7. Given $S = 126$, $r = 2$, and $l = 64$, find a and n.

Solution. By Formula 7,

$$126 = \frac{a - 2(64)}{1 - 2} = \frac{a - 128}{-1} = -a + 128. \tag{18}$$

Hence

$$a = 128 - 126 = 2 \text{ (answer)}. \tag{19}$$

Also, by Formula 5,

$$64 = (2)(2)^{n-1} = 2^{1+n-1} = 2^n. \tag{20}$$

But $2^6 = 64$, and hence

$$n = 6 \text{ (answer)}. \tag{21}$$

EXERCISE 63

1. Find the 5th term of the G.P.: 2, 4, 8, $\cdots$.
2. Find the 6th term of the G.P.: 1, 3, 9, $\cdots$.
3. Find the 7th term of the G.P.: 32, 16, 8, $\cdots$.
4. Find the 8th term of the G.P.: 1, -2, 4, $\cdots$.
5. Find the 9th term of the G.P.: 16, -8, 4, $\cdots$.
6. Insert 3 geometric means between 2 and 32.
7. Insert 4 geometric means between 32 and 1.
8. Insert 2 geometric means between 16 and -2.
9. Insert 4 geometric means between 1 and -243.
10. Insert 3 geometric means between 81 and 1.
11. Find the geometric means of 2 and 8.
12. Find the geometric means of 1 and 9.
13. Find the geometric means of -3 and -27.
14. Find the geometric means of -5 and -20.
15. Find the geometric means of $4a$ and $9a$.
16. If $a = 2$, $r = 2$, and $l = 16$, find n and S.
17. If $a = 32$, $l = -1$, and $r = -\frac{1}{2}$, find n and S.
18. If $a = 81$, $l = 1$, and $r = \frac{1}{3}$, find n and S.
19. If $a = 1$, $r = -2$, and $l = -32$, find n and S.
20. If $a = 16$, $r = \frac{1}{2}$, and $l = \frac{1}{2}$, find n and S.
21. If $S = 63$, $r = 2$, and $l = 32$, find n and a.
22. If $S = 121$, $r = 3$, and $l = 81$, find n and a.
23. If $a = 128$, $n = 8$, and $l = 1$, find r and S.
24. If $a = 16$, $n = 6$, and $l = \frac{1}{2}$, find r and S.

25. Find the sum of the areas of 5 squares if the side of the first is 12 inches, the side of the second is 6 inches, the next 3 inches, and so on to the last one.

26. A man buys 10 watermelons, paying 1¢ for the first, 2¢ for the second, 4¢ for the third and so on. Find the total cost.

27. Four men divide $5625 profit. Each one after the first receives half as much as the one before him. Find the amount each one receives.

28. An estate of \$11,808 is divided among 4 people so that each one after the first receives $\frac{4}{5}$ as much as the one before him. How much does each one receive?

29. Each person has 2 ancestors of the first preceding generation (parents), 2^2, or 4, of the second (grandparents), etc. How many ancestors has he in the twelfth preceding generation?

30. Assuming that a patriarch has 6 children (first following generation), that each of these has 6 children (second generation) and so on, find the total number of descendants of the patriarch through and including the fourth generation.

Chapter Fifteen

LOGARITHMS

15.1. *Remarks and definition*

The computation of a numerical quantity such as, for example, $\dfrac{\sqrt[3]{38.4(48.7)^5}}{\sqrt[7]{81.2}}$, would be a slow and tedious task were it not for the invention * called *logarithms*. This is a device which uses the laws of exponents to simplify calculations involving multiplication, division, *involution* [or evaluation of powers such as $(1.05)^{10}$], and *evolution* (or extraction of roots such as $\sqrt[4]{2321}$). Such laws are applicable because, as the following definition will show, a logarithm is essentially an exponent.

*Definition. The logarithm of the number **N** with respect to the base **b**, designated as "$\log_b N$," is the exponent which must be applied to **b** in order to produce **N**.*

15.2. *Exponential and logarithmic equations*

(1) $$10^2 = 100,$$

and

(2) $$\log_{10} 100 = 2,$$

the latter of which is read "the logarithm of 100 to the base 10 equals 2," are two forms of the same statement. The first is an *exponential equation*, and the second is a *logarithmic equation*. It should be noted that the exponent

* The inventor was *John Napier* (1550–1617), and the one who applied the invention to practical computation was *Henry Briggs* (1556–1631).

of 10 is 2 and the logarithm of 100 is 2. In both equations the base is 10. Similarly, if $b^x = N$, it follows that $\log_b N = x$.

It is very important to be able to express a logarithmic equation as an exponential one, and *vice versa*. For example, if $P = 10^x$, then $\log_{10} P = x$; if $\log_{10} MN = x + y$, then $MN = 10^{x+y}$.

15.3. *Four laws of logarithms*

Since logarithms are exponents, the laws applying to both are similar. We shall now derive the basic laws for logarithms.

1. $\log_a PQR = \log_a P + \log_a Q + \log_a R$.

In words, *the logarithm of a product is the sum of logarithms of its factors.*

For example,

$$\log_{10} (58.4)(73.6)(86.4) = \log_{10} 58.4 + \log_{10} 73.6 + \log_{10} 86.4.$$

Proof. Let $\log_a P = x$, $\log_a Q = y$, and $\log_a R = z$.

Then, $\quad P = a^x$, $Q = a^y$, and $R = a^z$.

Multiplying,

$$PQR = a^x a^y a^z = a^{x+y+z}. \text{ (Law 1, Art. 7.1)}$$

Changing the last equation to the logarithmic form, we have,

$$\log_a PQR = x + y + z = \log_a P + \log_a Q + \log_a R.$$

2. $\log_a \dfrac{P}{Q} = \log_a P - \log_a Q$.

In words, *the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

For example, by Law 2,

$$\begin{aligned}
&\log_{10} \frac{(58.4)(61.7)}{(15.6)(387)} \\
&\quad = \log_{10} [(58.4)(61.7)] - \log_{10} [(15.6)(387)] \\
&\quad = (\log_{10} 58.4 + \log_{10} 61.7) - (\log_{10} 15.6 + \log_{10} 387) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Law 1)} \\
&\quad = \log_{10} 58.4 + \log_{10} 61.7 - \log_{10} 15.6 - \log_{10} 387.
\end{aligned}$$

Proof. Let $\log_a P = x$ and $\log_a Q = y$.

Then, $P = a^x$ and $Q = a^y$.

Dividing $\dfrac{P}{Q} = \dfrac{a^x}{a^y} = a^{x-y}$. (Law 2, Art. 7.1)

Hence, $\log_a \dfrac{P}{Q} = x - y = \log_a P - \log_a Q.$

3. $\boldsymbol{\log_a P^n = n \log_a P.}$

In words, *the logarithm of the **n**th power of any number equals **n** times the logarithm of the number.*

For example, $\log_{10} (873.4)^3 = 3 \log_{10} 873.4.$

Proof. Let $\log_a P = x$, so that $P = a^x$.

Then, $P^n = (a^x)^n = a^{nx}$. (Law 5, Art. 7.1)

Hence, $\log_a P^n = nx = n \log_a P.$

4. $\boldsymbol{\log_a \sqrt[n]{P} = \dfrac{1}{n} \log_a P.}$

In words, *the logarithm of the **n**th root of a number equals* $\dfrac{1}{n}$ *times the logarithm of the number.*

For example, $\log_{10} \sqrt[3]{57.6} = \frac{1}{3} \log_{10} 57.6.$

Proof. Let $\log_a P = x$, so that $P = a^x$.

Then, $\sqrt[n]{P} = \sqrt[n]{a^x} = a^{\frac{x}{n}}$. (Def. 4, Art. 7.5)

Hence, $\log_a \sqrt[n]{P} = \dfrac{x}{n} = \dfrac{1}{n}(x) = \dfrac{1}{n} \log_a P.$

The following law is a corollary of Laws 3 and 4.

Corollary, or *Law* 5. $\boldsymbol{\log_a \sqrt[n]{M^m} = \dfrac{m}{n} \log_a M.}$

For example, $\log_{10} \sqrt[5]{(0.084)^3} = \frac{3}{5} \log_{10} (0.084).$

The proof is left to the student.

EXERCISE 64

Given $\log_{10} 2 = 0.30$ *and* $\log_{10} 3 = 0.48$, *use the laws of logarithms to find the logarithms of the numbers in problems* 1–16.

Examples.

(a) $\log_{10} 6 = \log_{10} (2 \cdot 3) = \log_{10} 2 + \log_{10} 3 = 0.30 + 0.48 = 0.78.$
(b) $\log_{10} 9 = \log_{10} 3^2 = 2 \log_{10} 3 = 2(0.48) = 0.96.$
(c) $\log_{10} 24 = \log_{10} 3 \cdot 2^3 = \log_{10} 3 + 3 \log_{10} 2 = 0.48 + 0.90 = 1.38.$
(d) $\log_{10} \frac{3}{4} = \log_{10} 3 - 2 \log_{10} 2 = 0.48 - 2(0.30) = -0.12.$

1. 12. **2.** 18. **3.** 27. **4.** 8. **5.** 16.
6. 32. **7.** $\frac{2}{3}$. **8.** $\frac{3}{2}$. **9.** $\frac{8}{3}$. **10.** $\frac{9}{4}$.
11. 5. HINT. $5 = \frac{10}{2}$. **12.** 25. **13.** 125.
14. .3 (or $\frac{3}{10}$). **15.** .2. **16.** .12 (or $\frac{3}{25}$).

Solve for x *the equations in problems* 17–34.

Examples.

(e) If $\log_2 x = 3$, $x = 2^3 = 8$.
(f) If $\log_2 8 = x$, $2^x = 8$, and hence $x = 3$.
(g) If $\log_x 8 = \frac{3}{2}$, $x^{\frac{3}{2}} = 8$, and hence $x = 4$.

17. $\log_3 x = 2$. **18.** $\log_4 x = -1$. **19.** $\log_8 x = \frac{2}{3}$.
20. $\log_2 x = -3$. **21.** $\log_5 x = 3$. **22.** $\log_9 x = -\frac{1}{2}$.
23. $\log_4 64 = x$. **24.** $\log_9 3 = x$. **25.** $\log_4 (\frac{1}{2}) = x$.
26. $\log_9 (\frac{1}{9}) = x$. **27.** $\log_8 (\frac{1}{4}) = x$. **28.** $\log_4 (8) = x$.
29. $\log_x 8 = 2$. **30.** $\log_x 9 = \frac{1}{2}$. **31.** $\log_x 3 = -\frac{1}{2}$.
32. $\log_x 8 = \frac{1}{3}$. **33.** $\log_x (\frac{4}{7}) = -1$. **34.** $\log_x 4 = -\frac{3}{2}$.

In each of problems 35–47 *state whether the equation is true or false. If it is false, change one member so that it will be true.*

Examples.

(h) $\log_a \left(\frac{A}{BC}\right) = \frac{\log_a A}{\log_a B + \log_a C}.$

False. Change right side to $\log_a A - \log_a B - \log_a C$.

(i) $\log_a\left(\frac{x^3y^2}{\sqrt[4]{z^3}}\right) = 3\log_a x + 2\log_a y - \frac{3}{4}\log_a z$. True.

(j) $\frac{x}{y} = \log_a x - \log_a y$.

False. Change left side to $\log_a\left(\frac{x}{y}\right)$.

35. $\log_a \sqrt{A} = \frac{1}{2}\log_a A$.

36. $\sqrt[3]{\log_a B} = \frac{1}{3}\log_a B$.

37. $(\log_a A)^{\frac{2}{3}} = \frac{2}{3}\log_a A$.

38. $\log_a (xy)^{10} = (\log_a x + \log_a y)^{10}$.

39. $\log_a\left(\frac{A}{B}\right)^3 = 3(\log_a A - \log_a B)$.

40. $\log_a \sqrt[4]{\frac{x^3}{y^3}} = \frac{3}{4}(\log_a x - \log_a y)$.

41. $\log_{10}\sqrt{\frac{xy}{z}} = \frac{1}{2}\left(\frac{\log_{10} x + \log_{10} y}{\log_{10} z}\right)$.

42. $\log_{10}\sqrt[3]{\frac{A^2}{B}} = (2\log_{10} A - \log_{10} B)^{\frac{1}{3}}$.

43. $\sqrt[5]{C} = \frac{1}{5}\log_a C$.

44. $10 = (\log_{10} 100)(\log_2 32)$.

45. $\log_{10}(\log_{10} 10) = 0$.

46. $a^{2\log_a x} = x^2$.

47. $\log_a a^3 = 3$.

15.4. *Systems of logarithms*

There are two standard systems of logarithms in general use in elementary mathematics. They are:

(a) The *Briggs* or *common* system, in which 10 is used as the base. This system is used extensively in numerical calculations, and will be used hereafter in this text. Thus, whenever the base is omitted, the base is to be understood as 10. For example, "log 58" will mean "$\log_{10} 58$."

(b) The *Naperian* or *natural* system, in which the base is an irrational number approximately equal to 2.718. This so-called *natural* base, usually designated by the letter e, is used extensively in advanced mathematics.

15.5. *Characteristics and mantissas*

In the following table, the two equations on the same line result one from the other, or, in other words, are corresponding exponential and logarithmic statements.

$$
\begin{array}{ll}
10^0 = 1; & \log 1 = 0. \\
10^1 = 10; & \log 10 = 1. \\
10^2 = 100; & \log 100 = 2. \\
10^3 = 1000; & \log 1000 = 3 \text{ (and so on).}
\end{array}
$$

Note that if we apply integral (whole number) exponents to 10 we produce only numbers having 1 and 0 as digits. It is clear that most numbers are omitted from this group. Hence it becomes necessary to use some exponents for 10 that contain decimal fractions in order to write all numbers as powers of 10.

For example, 25 lies between 10 and 100. The exponent that must be applied to 10 to produce 25 is therefore between 1 and 2. Actually

$$(1) \qquad 10^{1.3979} = 25 \text{ (nearly)},$$

or

$$(2) \qquad \log 25 = 1.3979.$$

That is, the exponent 1.3979, applied to the base 10, produces the number 25. This exponent consists of the integer 1 and the decimal fraction .3979. In general, the whole number part of any logarithm is called the *characteristic*, and the decimal part, if and when it is *positive*, is called the *mantissa*. For example, while $\log .25 = \log\left(\frac{25}{100}\right) = \log 25 - \log 100 = 1.3979 - 2 = -1 + .3979 = -.6021$, the mantissa of log .25 is .3979 rather than $-.6021$.

15.6. *Operations which leave the mantissa unchanged*

When a new number is obtained from a given number by moving the decimal point to the right or left, the first number

is in effect multiplied or divided by a power of 10. For example,

$$(1) \qquad 2500 = 25.00 \times 10^2,$$

and

$$(2) \qquad .025 = 25 \div 10^3.$$

From (1) and Law 1, we have

$$(3) \qquad \begin{aligned} \log 2500 &= \log 25 + \log 10^2 \\ &= \log 25 + 2 \log 10 \text{ (by Law 3)} \\ &= 1.3979 + 2(1) \\ &= 3.3979 \end{aligned}$$

From (2) and Law 2, it follows that

$$(4) \qquad \begin{aligned} \log .025 &= \log 25 - \log 10^3 \\ &= 1.3979 - 3(1) \\ &= -2 + .3979. \end{aligned}$$

In equations (3) and (4) we see *why* the mantissas of the logarithms of the three numbers: 25, 2500, and .025 are the same.

In general, when the decimal point in the number is moved n places to the right, the logarithm is increased by the integer n (or $\log 10^n$); and when it is moved n places to the left the logarithm is decreased by n. Hence in either case the mantissa remains unchanged. This very convenient result makes it possible to have compact tables of logarithms, since, for example, only one mantissa (.3979) need be recorded to apply to such various numbers as 25, 250, 2500, 2.5, .25, .025, etc.

15.7. *Rules about the characteristic*

Since $\log 10 = 1$ and $\log 100 = 2$, it is evident that 1 is the characteristic of the logarithm of 10 and of all numbers between 10 and 100; or, in other words, of all numbers

having two significant * digits to the left of the decimal point. Similarly, for numbers with three digits to the left of the decimal point, the characteristic is 2. In general, the following rule applies.

RULE 1. *The characteristic of the logarithm of a number greater than one is one less than the number of significant digits to the left of the decimal point.*

The logarithm of a number less than one is negative, but there is still a characteristic and a (positive) mantissa. For example, it was noted in (4) Art. 15.6 that $\log .025 = -2 + .3979$, with the characteristic -2 and mantissa .3979. For convenience, this result is usually written in the form

$$\log .025 = 8.3979 - 10, \tag{1}$$

since the right member of (1) is easy to use and preserves the positive mantissa.

Continuing the table in Art. 15.5 in the reverse direction, we have

$$10^{-1} = \frac{1}{10} = .1; \qquad \log .1 = -1.$$

$$10^{-2} = \frac{1}{10^2} = .01; \qquad \log .01 = -2.$$

$$10^{-3} = \frac{1}{10^3} = .001; \qquad \log .001 = -3 \text{ (and so on).}$$

With a little study of the table above we can arrive at a second rule about characteristics, namely,

RULE 2. *The characteristic of the logarithm of a number less than one is negative, and is equal numerically to one plus the number of zeros between the decimal point and the first non-zero digit on the right.*

* The *significant* digits in a number include the first non-zero digit, as read from left to right, plus all the digits that follow it. For example, each of the numbers 10.732 and 00030.21 has two significant digits to the left of the decimal point.

Examples.

THE NUMBER	THE CHARACTERISTIC OF THE LOGARITHM
.8003	$-(1 + 0) = -1$
.00207	$-(1 + 2) = -3$
.00000000004	$-(1 + 10) = -11$

In practice we may count the decimal point as well as the proper zeros, getting at once for the sum the correct negative characteristic. Thus the latter is obtained first as a negative integer, after which it is rewritten in the conventional form when the mantissa is known.

Example 1. $\log .00025 = -4 + .3979 = 6.3979 - 10.$

Example 2. $\log .000000000025 = -11 + .3979 = 9.3979 - 20.$

15.8.* *Scientific notation and a second method for determining characteristics*

It is possible to find the characteristic of the logarithm of a number N to base 10 by taking advantage of what is called the "scientific notation" method for representing N. For example, if 583.4 is written as $(5.834)(10^2)$; 0.000208 as $(2.08)(10^{-4})$; 3.08 as $(3.08)(10^0)$; 100,000 as $(1)(10^5)$; 0.3141 as $(3.141)(10^{-1})$; 0.000,000,000,108 as $(1.08)(10^{-10})$, etc., each of the numbers is said to be written in scientific notation. In other words, N is in scientific notation if it is represented as $(M)(10^k)$, where $1 \leqq M < 10$ (M is equal to or greater than 1 but less than 10).

For example, in 0.00208, $M = 2.08$, and $k = -3$.

RULE 3. *To determine the characteristic of a logarithm of a number to base* 10, *it is merely necessary to write (or think of) the given number in scientific notation, and the exponent of the* 10 *in this representation is the characteristic.*

* This article may be omitted without loss of continuity.

Example. Since, in scientific notation,

(a) $5870 = (5.87)(10^3)$;
(b) $3.4 = (3.4)(10^0)$;
(c) $0.0004108 = (4.108)(10^{-4})$;
(d) $0.01287 = (1.287)(10^{-2})$;
(e) $1{,}000{,}000 = (1)(10^6)$;
(f) $15 = (1.5)(10^1)$; and
(g) $0.000{,}000{,}000{,}012 = (1.2)(10^{-11})$;

it follows that the characteristics of the respective logarithms are the exponents of the 10 in the right members of the above equations. Thus the characteristics are (a) 3, (b) 0, (c) -4, (d) -2, (e) 6, (f) 1, and (g) -11.

EXERCISE 65 (ORAL)

Find the characteristic of the logarithms of each of the following numbers by both methods.

1. 36.	**2.** 407.	**3.** 8.	**4.** 2573.
5. 3.2.	**6.** 8.05.	**7.** 0.6.	**8.** 0.058.
9. 62.05.	**10.** 932.5.	**11.** 0.002.	**12.** 0.00015.
13. 2.75.	**14.** 0.235.	**15.** 0.056.	**16.** 0.0058.
17. 283.5.	**18.** 0.08276.	**19.** 3258	**20.** 865432.

15.9. *Finding the mantissa. Four-place tables*

The mantissa of the logarithm of a given number is found by use of a table, such as Table 3 at the end of the text. The first column on the left with the heading N contains the first two digits of the number. For example, if the number is 157, the digits 15 are found in the sixth row beneath N. The third digit, 7, is found on the line across the top of the table which contains N and the digits 0, 1, 2, etc. In the sixth row, under 7 and in line with the first two digits 15, we find the entry 1959. This means that the mantissa of the logarithm of 157, or .157, or 15,700, or .000157, etc. is .1959 (the decimal being omitted for brevity).

If the number whose logarithm is sought has two digits it is found in the column under N. Its mantissa will then be on the same line and in the column beneath the digit 0. If the number has one digit, we multiply it by 10 and use the mantissa of the product. For example, the mantissa of $\log 7 =$ the mantissa of $\log 70 = .8451$.

When we find the logarithm of a number we *first* write its characteristic by inspection (it is very important to get this habit) and *then* find the entry in the table which gives the digits in the mantissa.

Examples.

(a) $\log 4.3 = 0$ plus the mantissa. Here the student may be tempted to omit the characteristic; but this is a very bad practice, which often results in errors. *The characteristic should always be written first, even when it is zero.* In the table, in line with 43 and in the column under the digit 0, is the entry 6335. Hence, $\log 4.3 = 0.6335$.

(b) $\log 564 = 2$ plus the mantissa. In line with 56 in the table, and in the column under the digit 4, is the entry 7513, so that $\log 564 = 2.7513$.

(c) $\log 0.0108 = -2$ plus the mantissa. The latter, according to the entry in the table in line with 10 and under 8, is .0334. Writing the characteristic -2 in the recommended form, we have $\log 0.0108 = 8.0334 - 10$.

15.10. *Antilogarithms*

If $\log A = B$, then A is said to be the *antilogarithm of* B and may be found from the table by reversing the operation of finding the logarithm.

Examples.

(a) $\log N = 3.8102$. Find N.

We now look *inside* the table for the entry 8102. This is found in line with 64 and under the digit 6, so that N has

the digits 646 with the decimal point yet to be fixed. Looking next at the characteristic, 3, we see that there are four digits to the left of the decimal point in N. Hence $N = 6460$.

(*b*) $\log N = 7.3979 - 10$. Find N.

Inspection of the table shows that N has the digits 250. Since the characteristic is $7 - 10$ or -3, N is written with a decimal point followed by two zeros. Hence $N = .0025$.

(c) $\log N = 0.4133$. Find N.

The table shows that N has the digits 259. The characteristic 0 shows that N has one place to the left of the decimal point. Hence $N = 2.59$.

EXERCISE 66

Find the logarithm of each of the following numbers.

1. 183.	**2.** 347.	**3.** 526.	**4.** 235.
5. 621.	**6.** 420.	**7.** 360.	**8.** 924.
9. 32.5.	**10.** 52.6.	**11.** 3.72.	**12.** 8.28.
13. 0.062.	**14.** 0.735.	**15.** 0.0083.	**16.** 0.00091.
17. 0.0006.	**18.** 0.00584.	**19.** 0.000641.	**20.** 0.0000596.

Find N in each of problems 21–33.

21. $\log N = 1.0607$.	**22.** $\log N = 2.1303$.
23. $\log N = 1.3541$.	**24.** $\log N = 2.6335$.
25. $\log N = 1.6444$.	**26.** $\log N = 3.5933$.
27. $\log N = 9.4609 - 10$.	**28.** $\log N = 8.4983 - 10$.
29. $\log N = 7.3636 - 10$.	**30.** $\log N = 8.2455 - 10$.
31. $\log N = 0.7388$.	**32.** $\log N = 0.0043$.
33. $\log N = 6.0000 - 10$.	

15.11. *Interpolation*

When a given number, or a given mantissa, is not found directly in the table, it is necessary to use a process called *interpolation*, as illustrated in the following examples.

Example 1. Find log 2576.

Solution. Here the characteristic is 3. To obtain the mantissa, we observe that 2576 lies between the numbers 2570 and 2580, whose associated mantissas are the same as those for 257 and 258 respectively. Schematically the work is arranged as follows:

$$\begin{array}{ll} \text{Number} & \text{Mantissa digits} \\ 10\left[\begin{array}{l} 6\left[\begin{array}{l} 2570 \\ 2576 \end{array}\right. \\ \quad 2580 \end{array}\right. & \left.\begin{array}{r} 4099 \\ \\ 4116 \end{array}\right] 17 \text{ (the difference)} \end{array}$$

Now $4099 + \frac{6}{10}(17) = 4109.2$, but since four digits only are retained, the desired mantissa is .4109. Hence,

$$\log 2576 = 3.4109.$$

Evidently the process of interpolation would not have been changed if the decimal point had been differently placed in the digit sequence 2576. For example,

$$\log .0002576 = 6.4109 - 10.$$

It should be realized that interpolation uses the principle of proportional parts, which assumes that for a small change in the number there corresponds a proportional change in the mantissa. This is only approximately true, but in order to use interpolation this assumption will be accepted when the logarithm table does not contain directly the information desired.

Example 2. Find N if $\log N = 9.4177 - 10$.

Solution. The digits 4177 are not found among the mantissa digits inside the table; but they are seen to lie between the adjacent entries 4166 and 4183. The schematic arrangement follows:

$$\begin{array}{ll} \text{Number} & \text{Mantissa digits} \\ 10\left[\begin{array}{l} 2610 \\ \\ 2620 \end{array}\right. & \left.\begin{array}{l} \left.\begin{array}{l} 4166 \\ 4177 \end{array}\right] 11 \\ 4183 \end{array}\right] 17 \end{array}$$

Hence, the significant digits in N are

$$2610 + \tfrac{11}{17}(10) = 2616 \text{ (nearest four digit number).}$$

Thus, taking into account the characteristic of $\log N$, it follows that

$$N = 0.2616.$$

When first learning this process, the student should put the work in table form as illustrated in the examples. However, with enough practice and a clear understanding of the principle of proportional parts, he may often learn to interpolate mentally.

EXERCISE 67

Find the logarithm of each of the following numbers.

1. 2536.	**2.** 728.6.	**3.** 0.08352.
4. 35.27.	**5.** 839.6.	**6.** 0.005706.
7. 5.732.	**8.** 294.3.	**9.** 0.0007085.
10. 6.325×10^8.	**11.** 9.246×10^{-12}.	**12.** 2.318×10^{20}.

Find N in each of problems 13–24.

13. $\log N = 2.2226$.	**14.** $\log N = 6.3641$.
15. $\log N = 3.1145$.	**16.** $\log N = 7.2883 - 10$.
17. $\log N = 9.1463 - 10$.	**18.** $\log N = 0.7714$.
19. $\log N = 8.1633 - 10$.	**20.** $\log N = 7.4161 - 20$.
21. $\log N = 2.3146 - 10$.	**22.** $\log N = 0.4923 - 10$.
23. $\log N = 10.8143$.	**24.** $\log N = 15.0129$.

15.12. *Computation with logarithms*

It should be clearly understood that the usefulness of logarithms lies in the fact that they shorten computations involving multiplication, division, involution, and evolution (though not addition and subtraction). Furthermore, these operations could be performed in other ways, so that if time is not saved, the point in the use of logarithms is missed.

It is important, therefore, to take note of the details and practices which do save time. Chief among them are the following.

(a) The quantity to be evaluated should be written directly above the details of computation, and should be designated by some letter, as N.

(b) The computation should be outlined in full, *with the left members of all equations filled in before the logarithm table is used at all.*

(c) The logarithms should be arranged in neat column form, with the decimal points and the preceding equality signs in vertical lines.

The student should study carefully the illustrative examples below before attempting the problems in the subsequent exercise.

Example 1. Evaluate $\frac{324}{618}$.

Solution. Let $N = \frac{324}{618}$.

Then, $\log N = \log 324 - \log 618$ (by Law 2).

$$
\begin{aligned}
\log 324 &= 2. \\
\textit{Outline.}\quad \log 618 &= \underline{2.\qquad} \quad (-) \\
\log N &= \\
N &= \qquad\quad \text{(answer).}
\end{aligned}
$$

Note 1. In what follows we have repeated the outline to show how the mantissas should be entered. The student, however, should understand that after the complete outline is made, including all characteristics that it is possible to write down by inspection, he should make his entries from the table in *that same outline.*

Turning next to the table in the text, we fill in the outline as follows.

$$
\begin{aligned}
\log 324 &= 12.5105 - 10 \\
\log 618 &= \underline{\ \ 2.7910\qquad\ \ } \quad (-) \\
\log N &= \ \ 9.7195 - 10 \\
N &= .5242 \text{ (answer).}
\end{aligned}
$$

Note 2. Since in the details above we are obliged to subtract a larger from a smaller number it is helpful, as soon as this fact becomes apparent, to rewrite the characteristic of log 324 as 12 − 10 instead of 2. Remember that the decimal part of the logarithm is not the mantissa unless it is positive.

Note 3. In actual life many numbers used in computations are approximations with a limited number of significant digits. In this case the computed result should not have more significant digits than has any of the numbers which enter into the computation, and in the case above N should be written as .524. In the subsequent exercise, however, all numbers are assumed to be exact values rather than approximations.

In the following examples the values found in the table will be included along with the outline, though actually they should be filled in *after* the outline is made.

Example 2. Evaluate $\dfrac{(.02764)(3.268)^4}{\sqrt[3]{210{,}700}}$.

Solution. (Here interpolation *is* in order. The arrangement below suggests a compact, efficient way of recording all logarithms found, so that no details looked up are omitted in the final form of the result.)

$$\text{Let} \qquad N = \frac{(2764)(3.268)^4}{\sqrt[3]{210{,}700}}.$$

$$\begin{aligned}
\text{Then,}\ \log N &= \log 2764 + 4 \log 3.268 - \tfrac{1}{3}\log 210{,}700.\\
\log 2764 &= 3.4415\\
4\log 3.268 = 4(0.5142) &= \underline{2.0568} \quad (+)\\
\log \text{numerator} &= 5.4983\\
\tfrac{1}{3}\log 210{,}700 = \tfrac{1}{3}(5.3237) &= \underline{1.7746} \quad (-)\\
\log N &= 3.7237\\
N &= 5292 \text{ (answer)}.
\end{aligned}$$

Note 4. In the preceding example we found that the antilog of 3.7237 is halfway between 5292 and 5293. In cases

of this sort, we shall agree to use the *even* one of the two numbers between which the interpolated result lies.

Example 3. Evaluate $N = \sqrt[3]{0.01084} + (0.5136)^4$.

Solution. Let $A = \sqrt[3]{0.01084}$ and $B = (0.5136)^4$.

Then,

$$\log A = \tfrac{1}{3} \log 0.0184;$$
$$\log B = 4 \log (0.5136);$$

and

$$N = A + B.$$

Details.

$$\log A = \tfrac{1}{3}(8.0350 - 10) = \tfrac{1}{3}(28.0350 - 30) = 9.3450 - 10.$$
$$\log B = 4(9.7106 - 10) = 38.8424 - 40 = 8.8424 - 10.$$
$$\begin{aligned} B &= 0.06957 \\ A &= \underline{0.2213} \quad (+) \\ N &= 0.2909 \quad \text{(answer).} \end{aligned}$$

Note 5. Since we have no formula for the logarithm of a sum, it is necessary to find A and B separately.

Note 6. By a second method of finding $\log A$ we have $\frac{1}{3}(8.0350 - 10) = \frac{1}{3}(1.0350 - 3) = 0.3450 - 1$. However, the form indicated in the solution is customary.

Note 7. It is not necessary to take the final step indicated in the calculation of $\log B$, since the characteristic, -2, is obtained from $38 - 40$ as easily as from $8 - 10$.

Example 4. Evaluate $N = (0.008402)^{\frac{2}{7}}$.

Solution.
$$\begin{aligned} \log N &= \tfrac{2}{7} \log (0.008402) \\ &= \tfrac{2}{7}(7.9244 - 10) = \tfrac{1}{7}(15.8488 - 20) \\ &= \tfrac{1}{7}(65.8488 - 70) \\ &= 9.4070 - 10 \\ N &= 0.2553. \end{aligned}$$

Example 5. Evaluate $N = (27.84)^{-3} = \dfrac{1}{(27.84)^3}$.

Solution. $\log N = \log 1 - 3 \log (27.84)$.

$$\begin{aligned} \log 1 &= 10.0000 - 10 \text{ (or 0)} \\ 3 \log (27.84) = 3(1.4446) &= \underline{4.3338} \quad (-) \\ \log N &= 5.6662 - 10 \\ N &= 0.00004637. \end{aligned}$$

EXERCISE 68

By use of the four-place table of logarithms compute the value of each of the following quantities. (It is agreed * that $21 = 21.00$, $.031 = .03100$, etc.)

1. $\dfrac{(25)(76)}{83}$.
2. $\dfrac{(0.283)(41.6)}{388}$.
3. $\dfrac{(127.4)(32.6)}{0.0827}$.
4. $\dfrac{(576)(340)}{(228)^2}$.
5. $(213.4)(0.529)$.
6. $(3247)(6.97)(0.0844)$.
7. $3(41.62)^3(87.2)^2$.
8. $(0.732)^2$.
9. $(925.6)\sqrt{387.8}$.
10. $(0.00274)^{\frac{1}{3}}$.
11. $(0.009082)^{\frac{1}{2}}$.
12. $(327)^{\frac{2}{3}}$.
13. $(0.0528)^{\frac{3}{4}}$.
14. $(928)^{\frac{5}{2}}$.
15. $\sqrt[5]{(67.41)^2}$.
16. $\dfrac{1}{(387)^3}$.
17. $(5.76)^{-2}$.
18. $\sqrt{5298}$.
19. $\sqrt[3]{807.6}$.
20. $(8.37)^{-3}$.
21. $\sqrt[4]{3256}$.
22. $\sqrt[5]{8.362}$.
23. $\frac{4}{3}(3.142)(37.6)^3$.
24. $\sqrt{(23.8)(47.6)(39.8)(524)}$.
25. $\dfrac{1}{(58.4)(9.836)}$.
26. $\dfrac{1}{\sqrt{0.8846}}$.
27. $\dfrac{54.8}{(-1.06)^3}$. HINT. First find $-N$, which is positive.
28. $\sqrt[3]{\dfrac{(35.21)(568)}{-4.84}}$.
29. $\sqrt{\dfrac{(4.728)(175.3)}{(584)(755)}}$.
30. $\dfrac{(-4.76)(0.8138)}{(43.6)(-0,084)^2}$.

* *See* Note 3 in Article 15.12.

31. Given $S = \dfrac{3WPA^2}{43}$, find S if $W = 347.2$, $P = 6.7$, and $A = 0.7840$.

32. Given $T = \pi \sqrt{\dfrac{l}{g}}$, find T if $l = 733$, $g = 32.2$, and $\pi = 3.142$.

33. Given $M = \dfrac{b^2 + c^2 - a^2}{2bc}$, find $\log M$ if $a = 11.20$, $b = 43.43$, and $c = 48.38$.

34. $\dfrac{\sqrt[3]{10.20}}{0.45} - \sqrt[3]{41.3}$.

TABLES

ANSWERS

INDEX

Table 1. A LIST OF SYMBOLS

$+$	read "plus."
$-$	read "minus."
$a \times b$	read "a multiplied by b."
$a \div b$, or $\frac{a}{b}$, or a/b	read "a divided by b."
$=$	read "is equal to."
$\equiv$	read "is identical with."
$\neq$	read "is not equal to."
$<$	read "is less than."
$>$	read "is greater than."
$\leqq$	read "is less than or equal to."
$\geqq$	read "is greater than or equal to."
$a!$ or $\underline{\lvert a}$	read "factorial a or $1 \cdot 2 \cdot 3 \cdots a$."
( )	Parentheses
[ ]	Brackets
{ }	Braces
——	Vinculum
	These are signs of aggregation. They are used to collect algebraic expressions which are to be treated in operations as one algebraic expression.
$\log_a n$	read "logarithm of n to base a.
$\lvert b \rvert$	read "absolute value of b."
b^n	read "the nth power of b" or "b with the exponent n."
$\sqrt{a}$	read "square root of a."
$\sqrt[n]{a}$	read "nth root of a."
a_n	read "a subscript n" or "a sub n."
$f(x)$, $g(x)$, etc.	read "f of x," "g of x," or "the f function of x," etc.
∞	read "infinity"
$\propto$	read "varies as"

Table 2. SQUARES, CUBES, ROOTS

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
1	1	1.000	1	1.000	51	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	52	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	53	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	54	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	55	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	56	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	57	3,249	7.550	185,193	3.848
8	64	2.828	512	2.000	58	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	59	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	60	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	61	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	62	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	63	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	64	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	65	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	66	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	67	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	68	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	69	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	70	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	71	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	72	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	73	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	74	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	75	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	76	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	77	5,929	8.775	456,533	4.254
28	784	5.291	21,952	3.037	78	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	79	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	80	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	81	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	82	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	83	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	84	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	85	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	86	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	87	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	88	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	89	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	90	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	91	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	92	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	93	8,649	9.643	804,357	4.531
44	1,936	6.633	85,184	3.530	94	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	95	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	96	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	97	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	98	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	99	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	100	10,000	10.000	1,000,000	4.642
n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$

Table 3. COMMON LOGARITHMS

FOUR-PLACE LOGARITHMS

N	L. 0	1	2	3	4	5	6	7	8	9	Proportional Parts 1	2	3	4	5
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7
34	5315	5328	5340	5353	5366	5370	5391	5403	5416	5428	1	2	4	5	6
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4
N	L. 0	1	2	3	4	5	6	7	8	9	1	2	3	4	5

N	L. 0	1	2	3	4	5	6	7	8	9	Proportional Parts 1	2	3	4	5
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2
N	L. 0	1	2	3	4	5	6	7	8	9	1	2	3	4	5

ANSWERS

EXERCISE 1. *Page 2*

1. $2x$. **2.** $x - 4$. **3.** $\frac{x}{2} + 2$. **4.** $3x - 1$. **6.** $6x^2$. **7.** $3x$ miles; $(30 - 3x)$ miles. **8.** $3x + 4y$. **9.** $(20 - x - y)$ miles. **11.** $P = BH$. **12.** $T = \frac{BH}{2}$. **13.** $C = 2\pi R$. **14.** $I = Prt$. **16.** $2x + 3 = 3x - 2$. **17.** $\frac{1}{2}(x + 2y) = 2(y + 3x)$. **18.** $2(x + 1) - 2 = \frac{x + 4}{2} + 3$. **19.** x; $10 - x$. **21.** x; $3x$. **22.** x; $2x + 3$. **23.** x; $\frac{2}{3}x - 2$. **24.** $25x$.

EXERCISE 2. *Page 7*

1. $-15, -5, -4, -2, -1, -\frac{1}{2}, 0, \frac{2}{3}, 5, 6$. **2.** $0, -\frac{1}{2}, \frac{2}{3}, -1, -2, -4, \pm 5, 6, -15$. **3.** -4. **4.** -3. **6.** -10. **7.** -7. **8.** 12. **9.** -5. **11.** -7. **12.** -17. **13.** 32. **14.** -15. **16.** -4. **17.** -9. **18.** -7. **19.** 3. **21.** 3. **22.** -5. **23.** 10. **24.** -17. **26.** 31. **27.** -3. **28.** -12. **29.** -7. **31.** -13. **32.** 4. **33.** 3. **34.** 5. **36.** -17. **37.** 20. **38.** 31. **39.** 40. **41.** 8. **42.** -8. **43.** -40. **44.** 12. **46.** 0. **47.** 0. **48.** 0. **49.** 0. **51.** -42. **52.** 0. **53.** 0. **54.** 180. **56.** -30. **57.** -21. **58.** -1. **59.** -4. **61.** 0. **62.** -14. **63.** 5. **64.** -3. **66.** -10. **67.** -2. **68.** 6. **69.** -5. **71.** 5. **72.** 0. **73.** 0. **74.** Not a number. **76.** 0. **77.** 0. **78.** 0. **79.** Not a number. **81.** 0. **82.** Not a number. **83.** 1. **84.** 2. **86.** 1. **87.** 0.

EXERCISE 3. *Page 12*

1. (Sample answers) $2x^2yz^3$; $5x^2yz^3$; $-x^2yz^3$. **2.** (Sample answers): Monomials: $2x$; $3y$. Binomials: $3x+y$; $4m+3n$. Trinomials: $2x + 3y + 5$; $m^2 + 2mn + 5n^2$. **3.** $6x - 4$. **4.** $7-5x$. **6.** $5ax^2 - 5x + 9$. **7.** $4x^3 - 3x^2 - 2ax - 1$. **8.** $3x^3 - 6ax + 2$. **9.** $-3x^4 + x^2 + 4x - 2a - 1$. **11.** $x^3 + 3x^2 - 3x + 13$. **12.** $x^4 + 2x^2 - 2x + 3$. **21.** No. **22.** $2x - 10$. **23.** $9x - 1$. **24.** $ax^2 + 8x - 5$. **26.** $-2x^3 - 7x^2 + 4ax - 3$. **27.** $-x^3 - 4x^2$. **28.** $-7x^4 + 5x^2 - 2x - 2a + 1$. **29.** $-2x^4 + 3x^3 + 3x^2 + 2x - 4a + 3$. **31.** $-9x + 1$. **32.** $-ax^2 - 8x + 5$. **33.** $-ax^2 + x + 1$. **34.** $2x^3 + 7x^2 - 4ax + 3$. **36.** $7x^4 -$

$5x^2 + 2x + 2a - 1$. **37.** $2x^4 - 3x^3 - 3x^2 - 2x + 4a - 3$. **38.** $-a - b + d$. **39.** $7x - 2y - 1$. **41.** $-4a - b + 4$. **42.** $5a - 2b - c$. **43.** $-a + 2b + 4c + 1$. **44.** $-4x + 7$. **46.** $-x^2 + 3ax^2 + 3bx$. **47.** $-3x^2 - 2ax^2 + 2x$. **48.** $2x - (3a - b)$. **49.** $4x - (2a + 3y)$. **51.** $3a - (-4b + c + 1)$. **52.** $-3b - 5c$. **53.** $2x^3 - x^2 + 7x - 3$. **54.** $4a^3 - 10a^2 + 12ab + 3$.

EXERCISE 4. *Page 16*

1. $12a^4x^3$. **2.** $-15bx^4$. **3.** $24a^3x^5$. **4.** $-6ay^3$. **6.** $-3x$. **7.** $4xy^5$. **8.** $[(x + 1)(x - 1)](2x + 1) = (x^2 - 1)(2x + 1) = 2x^3 + x^2 - 2x - 1$. $(x + 1)[(x - 1)(2x + 1)] = (x + 1)(2x^2 - x - 1) = 2x^3 + x^2 - 2x - 1$. **9.** $12x^4 + 22x^3 - 2x^2 - 10x$. **11.** $9x^5 - 24x^4 + 13x^3 - 29x^2 + 18x - 3$. **12.** $2x^6 - 11x^5 + 19x^4 - 5x^3 - 17x^2 + 16x - 4$. **13.** $12x^6 + 32x^5 - 5x^4 - 47x^3 - x^2 + 11x - 2$. **14.** $x^7 - 2x^6 - 7x^5 + 21x^4 - 16x^3 - 9x^2 + 10x - 2$. **16.** $18x^6 - 6x^5 - 13x^4 + 9x^3 + 15x^2 + 2x - 4$. **17.** $3x^5 + 14x^4 + 15x^3 - 7x^2 - 8x + 3$. **18.** $8x^5 + 12x^4 - 32x^3 - 20x^2 + 48x - 16$. **19.** $3x^6 - 2x^5 - 42x^4 + 6x^3 + 45x^2 - 13x$. **21.** $18x^5 + 39x^4 - 4x^3 - 32x^2 - x + 6$. **22.** $3x^6 + \frac{11x}{2} - \frac{1}{2} - \frac{5}{2x}$. **23.** $x^4 - \frac{11x^3}{3} + \frac{7x^2}{3} - \frac{5}{3} + \frac{2}{3x}$. **24.** $x^3 - 2x^2 - 3$. **26.** $3x^2 - x - 2$. **27.** $x^3 + 4x^2 + 6x + 19 + \frac{55x - 21}{x^2 - 3x + 1}$. **28.** $3x^3 - 2x^2 + x + \frac{x^2 + 5x - 2}{2x^2 + 2x - 1}$. **29.** $2x^2 - 2x + 1 + \frac{5 - x}{3x^2 + x - 1}$. **31.** $2x - 3 + \frac{x + 2}{2x^2 + 3x - 2}$. **32.** $3x^2 - 17x + 48 + \frac{48 - 148x}{x^2 + 3x - 1}$. **33.** $x + 2 - \frac{1}{4x^2 - 3x + 1}$. **34.** $2x + 3 + \frac{3}{3x^2 + x - 2}$. **36.** $a^2 - ab + b^2$. **37.** $a^2 + ab + b^2$. **38.** $-3x^2 + 2ax + 5a^2$. **39.** $-3x^2 - x + 11$. **41.** $-3x + \frac{1}{3} + \frac{5}{3(1 - 3x)}$.

REVIEW EXERCISES FOR CHAPTER I. *Page 17*

1. $4x$. **2.** $\frac{3x}{4}$. **3.** $\frac{x}{5}$. **4.** $x - 7$ or $7 - x$. **6.** $3x - 9$. **7.** $5 - \frac{5x}{2}$. **8.** $10x + y$. **9.** $10y + x$. **11.** $2(10x + y) + 36$. **12.** $4x$ yds.; $10x$ yds.; $4x^2$ sq. yds. **13.** $(2x - 9)$ yds.; $(6x - 18)$ yds.; $(2x^2 - 9x)$ sq. yds. **14.** $600x$ ft. **16.** $7200(x + 1)$ ft. **17.** $V = \frac{4}{3}\pi R^3$. **18.** $A = \frac{h(a + b)}{2}$. **19.** $T = 0.10m + 0.40x$. **21.** $D = 2x - y$; $D = y - 2x$. **22.** $x = y + 10$. **23.** $I = 50x$. **24.** $I = .07R$. **26.** $s = 26{,}400x$. **27.** 24. **28.** -9. **29.** -9. **31.** -20. **32.** Not a number. **33.** 0. **34.** Not a

number. **36.** -18. **37.** -8. **38.** 30. **39.** -2. **41.** -32. **42.** $-\frac{1}{2}$. **43.** 76. **44.** -1. **46.** 9. **47.** $-8b^2$. **48.** 0. **49.** $2(a + b) = 2a + 2b$. **51.** $3y$. **52.** $8a^2 - 9ab + c$. **53.** $-8n^2 - mn + 10$. **54.** $6a - 10b + c$. **56.** $-x + 2y - 6z$. **57.** $2x^2 + 10xy - 17$. **58.** $-8ab - 11c^2 + 2d + 6$. **59.** $x + 4$. **61.** $-3a + 12b + 24$. **62.** $x + 5y + 8$. **63.** $-3x^2 - y - (2x - z + 3w)$. **64.** $a^2 - b^2 - (2a^2b^2 - 4a^4)$. **66.** $y - 5 - (x + 3z^2)$. **67.** $-6xy$. **68.** $21x^7y^6$. **69.** $-15m^3n^3$. **71.** $-10m^3n^2 + 15m^2n - 10mn^2$. **72.** $-3x^4 - 9x^3 - 6x^2 + 4x^2y + 7xy - y^2$. **73.** $4a^2 + 26ab - 14b^2$. **74.** $x^3 - x^2y + xy^2 - y^3$. **76.** $3x^5 + x^4y - 6x^2y - 2x^3y^2 + 4xy^2 + x^3y + x^2y^2 - 2y^2$. **77.** $x^2 + 7x + 12$. **78.** $x^2 - 7x + 10$. **79.** $x^2 + 4x - 21$. **81.** $-3a^2 + 3$. **82.** $2a^2 - 5a - 12$. **83.** $6a^2 - 19a + 10$. **84.** $-30a^2 - 4a + 2$. **86.** $4x^2 - y^2$. **87.** $-9x^2 + y^2$. **88.** $y^2 - 4x^2$. **89.** $-45x^4 + 5y^2$. **91.** $9 - 12x + 4x^2$. **92.** $-32 + 16x - 2x^2$. **93.** $-3x^2 - 12x - 12$. **94.** $100x^2 + 60x + 9$. **96.** $16x^2 + 40xy + 25y^2$. **97.** $-8y$. **98.** $25m$. **99:** $9ab^2$. **101.** $3x^2 + 2x + \dfrac{3}{2x - 1}\cdot$ **102.** $3x - 4y$. **103.** $\dfrac{2x^2}{3} + \dfrac{7x}{9} - \dfrac{52}{27} + \dfrac{25}{27(3x + 1)}$.

EXERCISE 5. *Page 23*

2. (Sample answers) (*a*) $2x^2 + x + 5$. (*b*) $2x^{\frac{1}{2}} - 3$. **3.** $10x^2 - 5ax - 5x$. **4.** $-3y^2 + 6xy^2$. **6.** $-6x^3 + 4ax^3 - 2bx^2 - 4x^2$. **7.** $-12ax - 6a^2$. **8.** $3p^3 + 3pq^2$. **9.** $x^2 + 10x + 25$. **11.** $-9x^2y^2 - 12xy - 4$. **12.** $-18m^4 - 24m^2n - 8n^2$. **13.** $4a^4 - 4a^2b + b^2$. **14.** $16m^4 - 24m^2x + 9x^2$. **16.** $4 - 4y^2 + y^4$. **17.** $9a^4c^2 - 3a^2c + \frac{1}{4}$. **18.** $9x^2 - 4y^2$. **19.** $4x^4 - 16y^4$. **21.** $\dfrac{a^2}{b^2} - \dfrac{b^2}{a^2}\cdot$ **22.** $x^2 + 2ax + a^2 - 4$. **23.** $9x^2 - 12x + 4 - 9y^2$. **24.** $x^2 + 2xy + y^2 - 4r^2 + 4rs - s^2$. **26.** $27a^3 + b^3$. **27.** $8p^6 + 27m^3p^3$. **28.** $a^3 - y^3$. **29.** $-24a^3 + 3b^3$. **31.** The square of a trinomial equals the sum of the squares of the terms plus twice the product of each pair of terms. **32.** $x^2 + y^2 + 4z^2 + 2xy + 4xz + 4yz$. **33.** $9x^2 + 4y^2 + z^2 - 12xy + 6xz - 4yz$. **34.** $4a^2 + b^2 + 9c^2 - 4ab - 12ac + 6bc$. **36.** $x(a + b + 1)$. **37.** $y(2ay - 1)$. **38.** $(2x + 1)^2$. **39.** $(x - 3y)^2$. **41.** $(2y - 1)(2y + 1)$. **42.** $(a + b - c - d)(a + b + c + d)$. **43.** $(1 - 2x + y)(1 + 2x - y)$. **44.** $(2 - 3y)(4 + 6y + 9y^2)$. **46.** $(3x^2 - 2y^2)(9x^4 + 6x^2y^2 + 4y^4)$. **47.** $(a + b - 2)(a^2 + 2ab + b^2 + 2a + 2b + 4)$. **48.** $(8 - x - 2y)(8 + x + 2y)$. **49.** $(x + y)(x + y - 1)$. **51.** $(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$. **52.** $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$. **53.** $(4a - 5b)(4a + 5b)$. **54.** $(x - y - a - b)(x - y + a + b)$. **56.** $(k + 1 - x + y)(k + 1 + x - y)$.

57. $25x^3y(2xy-1)(2xy+1)$. **58.** $12(a^4b^5-3)$. **59.** $16xy^3(2xy-1)(2xy+1)$.

EXERCISE 6. *Page 26*

1. $(x+1)^2$. **2.** $(2x+3)^2$. **3.** $(4x+5)^2$. **4.** $(a-5)^2$. **6.** $(x-2)^2$. **7.** $(3x-2)^2$. **8.** $(5x-4)^2$. **9.** $(2m-1)^2$. **11.** $(x-3)(x-2)$. **12.** $(x-3)(x+1)$. **13.** $(x+2)(x+1)$. **14.** $(x-6)(x+1)$. **16.** $(x+6)(x-1)$. **17.** $(x-2)(x-1)$. **18.** $(x-7)(x+1)$. **19.** $(x+2)(x-1)$. **21.** $(x-6)(x-5)$. **22.** $(x-12)(x+1)$. **23.** $(x+12)(x-1)$. **24.** $(x-5)(x+1)$. **26.** $(x-7)(x-3)$. **27.** $(x+8)(x+4)$. **28.** $(x+13)(x-1)$. **29.** $(x-13)(x+1)$. **31.** $(2x+1)(x+2)$. **32.** $(2x+1)(x-2)$. **33.** $(3x-1)(x+2)$. **34.** $(5x+2)(x-1)$. **36.** $(3x-2)(2x+3)$. **37.** $(3x+2)(2x-3)$. **38.** $(5x-3)(2x-1)$. **39.** $(5x-3)(2x+1)$. **41.** $(3x+5)(2x-1)$. **42.** $(5x-3)(3x-2)$. **43.** $(3x-2)(3x+1)$. **44.** $3(x+2)(3x-1)$. **46.** $(8x-3)(3x+2)$. **47.** $(9x+4)(3x-2)$. **48.** $(10x+3)(2x+1)$. **49.** $(12x-5)(3x+2)$.

EXERCISE 7. *Page 28*

1. $(a+b)(x-y)$. **2.** $(b-c)(x+y)$. **3.** $(x+y)(a+b)$. **4.** $(m-n)(k-1)$. **6.** $(x+y)(a+b)$. **7.** $(x-y)(b+c)$. **8.** $(x-2)(x^2+1)$. **9.** $(x-1)(x+1)(2x+3)$. **11.** $(x+y+1)(x-y)$. **12.** $(x-y)(x+y-1)$. **13.** $(2x+3y+1)(2x-3y)$. **14.** $(3x+4y)(3x-4y-1)$. **16.** $(2x-y+1)(2x+y-1)$. **17.** $(x+y-3)(x+y+3)$. **18.** $(4-x+y)(4+x-y)$. **19.** $(x-y+5)(x-y-5)$. **21.** $(y+1-x)(y+1+x)$. **22.** $(2x-1-y)(2x-1+y)$. **23.** $(x-3-y)(x-3+y)$. **24.** $(x-a-b+1)(x-a+b-1)$. **26.** $(x^4-x^2+1)(x^2-x+1)(x^2+x+1)$. **27.** $(x^2+x-1)(x^2-x-1)$. **28.** $(x^2-x+2)(x^2+x+2)$. **29.** $(x^2+3-x)(x^2+3+x)$. **31.** $(x^2+x-3)(x^2-x-3)$. **32.** $(x^2-2x+3)(x^2+2x+3)$. **33.** $(x^2+4-2x)(x^2+4+2x)$. **34.** $(2x^2-x+1)(2x^2+x+1)$. **36.** $(x+y)(1-x+y)$. **37.** $(2x-y+2)(2x+y-2)$. **38.** $(2x^2-2x+1)(2x^2+2x+1)$. **39.** $(5x^2-3x+1)(5x^2+3x+1)$. **41.** $(3x-1)(3x+1)(x+1)(x-1)$. **42.** $(3x-2)(3x+2)(x-2)(x+2)$.

EXERCISE 8. *Page 30*

1. 3; 180. **2.** 6; 72. **3.** 9; 108. **4.** 1; 252. **6.** 15; 450. **7.** 16; 480. **8.** 18; 540. **9.** 13; 3640. **11.** $5a^2x^2$; $30a^4x^4$. **12.** $3x^2y^2$; $180x^4y^4$.

13. $x - y$; $x^3 - xy^2$. **14.** $x + y$; $x^3 - xy^2$. **16.** $x + y$; $x^2(x - y)(x + y)^2$. **17.** $x - 2$; $(x - 2)(x + 2)(x - 1)(x - 3)$. **18.** $2x - 1$; $(2x - 1)(x - 1)(x + 3)(2x + 1)$. **19.** $3x - 1$; $(3x - 1)(x + 2)(x - 1)(x + 1)$. **21.** $3x - 1$; $(3x - 1)(2x + 1)(x - 1)(2x - 1)$. **22.** $5x + 4$; $(5x + 4)(x - 1)(3x - 2)(2x - 3)$. **23.** $2y - x$; $x^2(2y - x)^3$. **24.** $(x - y)^2$; $y^4(x - y)^3$.

REVIEW EXERCISES FOR CHAPTER II. *Page 31*

1. $4y^2(y - 3)$. **2.** $3b^2(a + 2b)$. **3.** $mn(m - n)$. **4.** $ab(3a^2 + 5b - 1)$. **6.** $(x - 3y)(y + 2)$. **7.** $(c - 2a)(3a - b)$. **8.** $(2m - n)(3 - x^2 - y^2)$. **9.** $(m + c)(x + y)$. **11.** $(x - 1)^2(x + 1)$. **12.** $(3b - 1)(2a + 1)$. **13.** $(x - 4y)(x^2 - 2y^2)$. **14.** $(2q - 3)(3p + 2)$. **16.** $(x + 3y)(x - 3y + 1)$. **17.** $(y - 4)(y + 3)$. **18.** $(2x + 1)(x + 2)$. **19.** $(x + 7)(x + 3)$. **21.** $(x + 4y)(x - 3y)$. **22.** $(p^2 + 18)(p^2 - 2)$. **23.** $(4a + 1)(3a - 1)$. **24.** $(5r + 3s)(5r - s)$. **26.** $(3x - 1)(4x + 25)$. **27.** $(10 - x)(2 + x)$. **28.** $(x^4 + 2y^4)^2$. **29.** $(2c - d)^2$. **31.** $(2 - 5x)^2$. **32.** $(3x^2 - yz)^2$. **33.** $(2x - 11y)(2x + 11y)$. **34.** $(3 - x)(3 + x)(9 + x^2)$. **36.** $(3x - y - 3)(3x - y + 3)$. **37.** $(x - 2y - z)(x + 2y + z)$. **38.** $(x + 2y - 3m + 3n)(x + 2y + 3m - 3n)$. **39.** $(1 - x - y)(1 + x + y)$. **41.** $(3 - a + 4b)(3 + a - 4b)$. **42.** $(2x - a - b + y)(2x - a + b - y)$. **43.** $(p + q - a)(p + q + a)$. **44.** $(x + 2)(x^2 - 2x + 4)$. **46.** $(ay + 4)(a^2y^2 - 4ay + 16)$. **47.** $(x^2 + a^2)(x^4 - a^2x^2 + a^4)$ **48.** $(2x - 3)(4x^2 + 3)$. **49.** $(3y + a + 2b)(9y^2 - 3ay - 6by + a^2 + 4ab + 4b^2)$. **51.** $(x^2 - 4x + 8)(x^2 + 4x + 8)$. **52.** $(x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$. **53.** $(x^2 - xy + y^2)(x^2 + xy + y^2)$. **54.** $(x^2 - 4 - 4x)(x^2 - 4 + 4x)$. **56.** $(a^2 - 8b^2)(a^2 - 2b^2)$. **57.** $r^2(2y - 5)(2y + 5)$. **58.** $y^2(y - 2)^2(y + 2)^2$. **59.** $(x - a)(x + a)(x^2 + a^2)$. **61.** $(a + 2)(a - 2)(a + 1)$. **62.** $(y - a)^2(y + a)^2$. **63.** $(m - 2)(n + 1)(n - 1)$. **64.** $(x + 3m)(x^2 - 3mx + 9m^2)$. **66.** $(2x - 3y - 6)(2x + 3y - 2)$. **67.** $a(3x + y)(x - y)$. **68.** $8rm(rx - 2 - 6r^2m^2)$. **69.** $(1 - b)(1 + b)(x + y)$. **71.** $(pq - 1)(pq + 1)$. **72.** $(2x + 7c)^2$. **73.** $(x - a - 2)(x^2 + ax + 2x + a^2 + 4a + 4)$. **74.** $(2x + 1)(x - 1)^2$.

EXERCISE 9. *Page 37*

1. (*a*) 1; 3. (*b*) 3; 1. (*c*) 2; 2. **2.** In case (*b*). **3.** $\frac{a}{b + 1}$.

Canceling not permissible in problems **4, 5, 7, 9, 10,** and **12.** **6.** $\frac{x - 1}{3}$.

8. a. **11.** $\frac{1}{3(x + y)}$. **13.** $\frac{x - y}{a}$. **18.** $\frac{1}{3}$. **19.** $\frac{2}{3}$. **21.** $\frac{7}{5}$. **22.** $\frac{x^2 + x + 1}{x + 1}$.

23. $\frac{2x}{3}$. 24. $\frac{x}{2}$. 26. $\frac{6(x+y)}{5}$. 27. $\frac{5(y+1)}{2}$. 28. $\frac{7y(x+3)}{3}$. 29. $\frac{x-4}{x+4}$.
31. $\frac{x+3}{x-3}$. 32. $\frac{x-5}{x-3}$. 33. $\frac{x-2}{x+2}$. 34. $\frac{x-2}{x-1}$. 36. $\frac{3x-2}{x-1}$. 37. $\frac{3x-1}{2x-1}$.
38. $\frac{3x-1}{2x+1}$. 39. $\frac{3x+2}{2x+3}$. 41. $\frac{x^2-4}{x^2-9}$. 42. $\frac{x^2+1}{x-x^3}$. 43. $\frac{x^2+2}{3-x^2}$.
44. $\frac{5c^2-10cd+5d^2-x}{c+d}$. 46. $\frac{x^2+2xy+2y^2}{5}$. 47. $\frac{x+y+8}{x+y-3}$.
48. $\frac{x-4y}{x^2+xy+y^2}$. 49. $\frac{a^2-2ab+4b^2+a-2b}{a-2b}$.

EXERCISE 10. *Page 40*

1. -1. 2. 1. 3. -1. 4. 1. 6. 1. 7. -1. 8. -1. 9. -1.
11. $\frac{4x+1}{4}$. 12. $\frac{x+2}{6}$. 13. $\frac{8x-5}{5}$. 14. $\frac{4x-1}{8}$. 16. $\frac{3x+1}{x+1}$.
17. $\frac{x+1}{2x-1}$. 18. $\frac{7x+3}{x-9}$. 19. $\frac{5x-8}{x+1}$. 21. 0.

EXERCISE 11. *Page 43*

1. $\frac{5}{6}$. 2. $\frac{14}{15}$. 3. $\frac{19}{20}$. 4. $\frac{20}{21}$. 6. $\frac{61}{60}$. 7. $\frac{1}{280}$. 8. $\frac{41}{40}$. 9. $\frac{47}{36}$.
11. $\frac{x}{2-x}$. 12. $\frac{4x-7}{2x-3}$. 13. $\frac{14x-3}{12}$. 14. $\frac{21x+40}{60}$. 16. $\frac{20x-15}{18}$.
17. $\frac{9x-30}{20}$. 18. $\frac{6-x}{10}$. 19. $\frac{12-5x-2x^2}{6x}$. 21. $\frac{9+6x-2x^2}{6x}$.
22. $\frac{3x^2+15x-3}{5x}$. 23. $\frac{6x^2-20x+7}{4x}$. 24. $\frac{6x^2-28x+3}{7x}$.
26. $\frac{5x^2+2x+15}{x^2-9}$. 27. $\frac{7x+5}{2x-1}$. 28. $\frac{5x^2-3x-19}{6x^2+5x-6}$.
29. $\frac{2x^2+31x-43}{6x^2+x-12}$. 31. $\frac{14x^2+14x+1}{4x^2-1}$. 32. $\frac{x^2-13x}{x^2+4x+3}$.
33. $\frac{8x^2-7x+12}{3x^2-2x-1}$. 34. $\frac{3x^2-18x-28}{6x^2+5x-6}$. 36. $\frac{6x^2+3x-41}{6x^2+13x-5}$.
37. $\frac{9x-7y}{3x-2y}$. 38. $\frac{8x-32y}{7y-5x}$. 39. $\frac{23x^2-49xy+15y^2}{6x^2-5xy+y^2}$.
41. $\frac{8x+2y}{2y-3x}$. 42. $\frac{32x-6y}{y-7x}$. 43. $\frac{x^3+3x^2-4xy+8y^2+y^3}{x^3+y^3}$.
44. $\frac{x^3+3x^2+9xy+9y^2-y^3}{x^3-y^3}$. 46. $\frac{33-10x}{15x}$. 47. $\frac{9x-19}{15x}$.
48. $\frac{9x^2-10x+10}{6x}$. 49. $\frac{26x-15-8x^2}{12x}$.

EXERCISE 12. *Page 45*

1. 5. **2.** $\frac{14}{5}$. **3.** $\frac{8}{3}$. **4.** $-\frac{8}{25}$ **6.** $-\frac{3}{14}$. **7.** $\frac{3}{14}$. **8.** -3. **9.** $\frac{3}{2}$.

11. $\dfrac{2x^2 - 2x}{x + 2}$ **12.** $-2x - 3$. **13.** $\dfrac{15x^2 - 4x - 4}{3x - 1}$. **14.** $6x - 14$.

16. $6x - 9$. **17.** $39x + 9$. **18.** $\dfrac{17x^2 - 3x}{3}$. **19.** $\dfrac{2x - 13x^2}{7}$.

21. $\dfrac{21x^2 - 9x}{2x - 1}$. **22.** $10x$. **23.** $2x$. **24.** 4. **26.** 3.

27. $\dfrac{45x^2 + 15x - 30}{2x^2 - 13x + 15}$. **28.** $\dfrac{3x - 1}{x}$. **29.** $\dfrac{3x - 2}{x}$. **31.** $\dfrac{7 - 5x}{2x}$.

32. $-\dfrac{4x + 3}{5x}$. **33.** $\dfrac{6x^2 - 13x + 6}{x^2 - 9}$. **34.** $\dfrac{10x^2 - x - 21}{x^2 + 2x - 35}$. **36.** $\dfrac{x - 3}{2x + 6}$.

EXERCISE 12a. *Page 46*

1. $\dfrac{1}{3x - 4}$. **2.** $\dfrac{3}{5x + 3}$. **3.** $\dfrac{4}{7x + 10}$. **4.** $\dfrac{2}{2x + 3}$. **6.** $-\dfrac{x + 2}{x^2 + 4}$.

7. $\dfrac{x - 2}{x^2 + x - 2}$. **8.** $\dfrac{2x - 1}{2x^2 - x - 3}$. **9.** $\dfrac{12x^2 - 7x - 12}{36x^3 + 9x^2 - 43x + 12}$.

11. $\dfrac{3x^2 + 7x - 6}{x - 2}$. **12.** $\dfrac{2x^2 + x - 10}{3x - 2}$. **13.** $\dfrac{10x^2 - 31x + 15}{x + 4}$.

14. $5 - 3x$. **16.** $\dfrac{6x^2 - 5x - 6}{x - 1}$. **17.** $\dfrac{6x^2 - 11x - 2}{3x - 2}$. **18.** $\dfrac{x^2 + 2x + 1}{3x^2 - 2x - 1}$.

19. $\dfrac{6x^3 + 7x^2 - 9x + 2}{2x - 3}$. **21.** $\dfrac{1}{2 - x}$. **22.** -2. **23.** x^2. **24.** $\dfrac{3y^2 + 3y}{2}$.

26. $\dfrac{9x + 6}{2}$. **27.** $8x^2 - 12x + 4$. **28.** $7(x^2 + 2x - 1)$. **29.** 2.

31. $\dfrac{9x^2 + 6x + 1}{x^2 + 2x + 1}$. **32.** $\dfrac{(y - 3x)(x + y)^2}{(y + 3x)(y - x)(x + 2y)}$. **33.** $\dfrac{2x + y}{4x - 2y}$.

34. $\dfrac{(3x - 2y)(x - 2)}{(y - x)(x + 2)}$. **36.** $\dfrac{y - 1}{y - 2}$.

EXERCISE 13. *Page 49*

1. $\dfrac{y}{x}$. **2.** $\dfrac{xy + 1}{xy - 1}$. **3.** $\dfrac{x - 1}{x + 1}$. **4.** $\dfrac{1 + x + y}{1 - x - y}$. **6.** -1. **7.** 1.

8. -1. **9.** $x + y$. **11.** $\dfrac{(3x - 2y)(x - 2)}{(x - y)(x + 2)}$. **12.** $\dfrac{(x - y)(2x - y)}{(2x + y)(x - 2y)}$.

13. $\dfrac{2x + y}{4y - 2x}$. **14.** $-\dfrac{(x + y)^2}{(x - y)^2}$. **16.** $\dfrac{x^2 + xy + y^2}{x^2 - y^2}$. **17.** $\dfrac{3y - 3x}{10}$.

18. -1. **19.** $\dfrac{5y - 3x}{5y + 2x}$. **21.** $\dfrac{y - 2x - x^2 + 2xy - y^2}{2y - x + x^2 - 2xy + y^2}$. **22.** $\dfrac{x - 1}{x - 4}$.

23. $\dfrac{x^2-3x}{x+1}$. 24. $\dfrac{6x^2-x-4}{2x^2-x-1}$. 26. $\dfrac{5x+2}{x+1}$. 27. $\dfrac{1}{1-x}$.

28. $\dfrac{6x^2+4x+1}{(1-2x)(x+1)}$. 29. $\dfrac{x+3}{2x+1}$.

REVIEW EXERCISES FOR CHAPTER III. *Page 51*

1. $\dfrac{4a-2b}{a-3b}$. 2. $\dfrac{3x-2y}{9x^2+6xy+4y^2}$. 3. $\dfrac{x^2+4xy+4y^2-xz-2yz+z^2}{x+2y-z}$.

4. $\dfrac{3n+c}{3n-c}$. 6. $\dfrac{4x^2-4xy+y^2-5}{2x+y}$. 7. $-\dfrac{3x+23}{24}$. 8. $\dfrac{7a-15y}{12y}$.

9. $\dfrac{9x^3-3x^2y+8x^2y^2-4xy-12xy^2-3y^2}{12x^2y^2}$. 11. $\dfrac{5x^2-xy+x+y}{(x-y)^2(x+y)}$.

12. $\dfrac{a+b}{(a-c)(b-c)}$. 13. $\dfrac{2x^2-8x+2}{3-x}$. 14. $\dfrac{5x^2-4x+11}{12x^2-12}$. 16. $\dfrac{x^2-1}{xy-1}$.

17. $\dfrac{3a^2-6ab-b^2}{3a-b}$. 18. $\dfrac{2(x+y)}{z(x-y)}$. 19. $\dfrac{ab}{3b-1}$. 21. $\dfrac{x^2-1}{x}$.

22. $\dfrac{a^2+3ab+2b^2}{a^4+4b^4}$. 23. $\dfrac{(a^3-8)(a+2)}{8}$. 24. $\dfrac{4x^2+2xy}{y^2}$.

26. $\dfrac{(x^2+1)(x^2-4)}{x^2-1}$. 27. $\dfrac{2x+1}{2}$. 28. $\dfrac{(2a-5)(a-1)}{6-3a-a^2}$.

29. $\dfrac{(y-x)(3+xy-x^2)}{y-x+2}$. 31. $\dfrac{x+y}{x-y}$. 32. $\dfrac{a+b}{a^2+ab+b^2}$.

33. $\dfrac{5a+5b-3}{2a+2b-1}$. 34. $\dfrac{6ab-3b-3a-ab}{2ab-b-a}$. 36. $\dfrac{(a^2+5)(2a^2+7)}{(a^2+4)^2}$.

37. $-\dfrac{(x+3)(x+5)}{(x+2)^2}$. 38. $-\dfrac{3u^3+6u^2+2u}{(u+2)^2}$.

39. $\dfrac{2ac}{(b-a-c)[(a-c)^2-b^2]}$. 41. $-\dfrac{(a+b)^3}{a^3b}$. 42. $\dfrac{4ab+a-2b}{a^2-b^2}$.

43. $\dfrac{m^3-mn^2-2n^3}{n^3}$. 44. $\dfrac{b^3c}{a}$.

EXERCISE 14. *Page 59*

The equations in problems **3**, **5**, **7**, **9** and **11** are identities. **1.** 5.

2. -4. **4.** 2. **6.** 2. **8.** -1. **12.** 5. **13.** $\dfrac{-3}{8}$. **14.** $\frac{19}{2}$. **16.** $\frac{10}{3}$.

17. $\frac{7}{2}$. **18.** $\frac{73}{44}$. **19.** $\frac{10}{3}$. **21.** $\frac{5}{2}$. **22.** $\frac{7}{2}$. **23.** -4. **24.** $\frac{22}{9}$.

26. $\frac{43}{7}$. **27.** $\dfrac{-1}{8}$. **28.** $\frac{3}{5}$. **29.** -1. **31.** -1. **32.** $\frac{9}{46}$. **33.** $\frac{5}{2}$.

34. -9. **36.** 3. **37.** 4. **38.** 4; 8. **39.** 9; 3. **41.** \$500; \$1000.

42. 6 P.M.

XERCISE 15. *Page 61*

1. No root. **2.** 1. **3.** No root. **4.** 2. **6.** 4. **7.** $-\frac{10}{13}$. **8.** $\frac{1}{2}$.
. $\frac{41}{6}$. **11.** 9. **12.** 1. **13.** No root. **14.** $-\frac{1}{15}$. **16.** $-\frac{5}{12}$. **17.** -1.
8. $-\frac{57}{14}$. **19.** $-\frac{4}{3}$. **21.** 1. **22.** -2. **23.** 2. **24.** $\frac{1}{2}$. **26.** 10.
7. $-\frac{8}{11}$. **28.** 1. **29.** $-\frac{25}{2}$. **31.** $-\frac{2}{5}$. **32.** $\frac{1}{6}$. **33.** No root.

XERCISE 16. *Page 65*

1. $\frac{6b + 2a}{6 - bc}$. **2.** $\frac{6bd}{5(4a + c - 4d)}$. **3.** $\frac{a + d}{2c}$. **4.** b. **6.** $-d$.
. $-c$. **8.** $\frac{adm + adn + bc}{cm + cn + d}$. **9.** $-\frac{na}{b}$. **11.** $\frac{-bm}{n}$. **12.** $-\frac{an}{m}$.
3. $\frac{6br + 2bn}{r - 3n}$. **14.** $\frac{ar - 3an}{6r + 2n}$. **16.** $\frac{r(a - 6b)}{3a + 2b}$. **17.** $\frac{2r}{1 + 6rn - 4rm}$.
8. $\frac{2m - 2mn - 2m^2}{3m + 3n + 3}$. **19.** $\frac{3a}{2 - b}$. **21.** $l - nd + d$; $\frac{d - a + l}{d}$; $\frac{l - a}{n - 1}$.
2. $\frac{En - IRn}{I}$; $\frac{En - IR}{In}$; $\frac{Ir}{E - Ir}$. **23.** $\frac{fp}{p - f}$; $\frac{fq}{q - f}$; $\frac{pq}{p + q}$.
4. $\frac{RR_2}{R_2 - R}$; $\frac{RR_1}{R_1 - R}$; $\frac{R_1R_2}{R_1 + R_2}$. **26.** $\frac{aM}{v - a}$; $\frac{vm - am}{a}$; $\frac{aM + am}{m}$.
7. $\frac{Sr - S + a}{r}$; $rl + S - Sr$, $\frac{S - a}{S - l}$. **28.** $\frac{Fd^2}{KM}$; $\frac{Fd^2}{Km}$. **29.** $\frac{v - v_0}{g}$;
$- gt$; $\frac{v - v_0}{t}$. **31.** $\frac{2A}{a + b}$; $\frac{2A - ha}{h}$. **32.** $\frac{br^2 + V}{r^2}$. **33.** $\frac{aW_2 - W_1}{a - 1}$;
$\frac{V_1 + aW - W}{a}$. **34.** $(a + b)$ years. **36.** $2a + 20$. **37.** $(5x + 25)$ cents.
8. $\frac{2a}{3}$. **39.** $\frac{ac + bc}{2}$ miles. **41.** $\frac{100a}{c}$ dozens.

XERCISE 17. *Page 67*

1. 15. **2.** 23. **3.** 12. **4.** 6; 9. **6.** 9; 12. **7.** 3. **8.** 5.
. 12. **11.** $\frac{3}{5}$. **12.** 17. **13.** $\frac{2}{7}$.

XERCISE 18. *Page 70*

1. $\frac{21}{5}$ ft.; $\frac{14}{5}$ ft. **2.** $\frac{5}{2}$ ft.; $\frac{3}{2}$ ft. **3.** 7 in.; 10 in. **4.** 6 in.;
in.; 12 in. **6.** 10 in. × 10 in.; 15 in. × 10 in. **7.** 3 in.;
in. **8.** 4 in.; 8 in. **9.** 10 in. × 6 in.; 6 in. × 6 in. **11.** 1 in.;
in. **12.** 50; 40. **13.** 30°; 60°; 90°. **14.** 40°; 100°. **16.** 20°;
0°; 100°.

EXERCISE 19. *Page 72*

2. 15 hrs. after 12 o'clock. **3.** $\frac{90}{17}$. **4.** $\frac{40}{17}$. **6.** $\frac{440}{17}$. **7.** 6 hrs. after 1st car starts. **8.** $\frac{160}{3}$ mph. **9.** 6 mi.; 3 hrs.; 2 hrs. **12.** 10 mph. **13.** 20 mph. **14.** $\frac{39}{5}$ mph. **16.** 2 mph. **17.** 100 mph. **18.** 525 mph.

EXERCISE 20. *Page 76*

1. \$30. **2.** 4%. **3.** 4%. **4.** $\frac{10}{3}$%. **6.** 2 yrs. **7.** \$5000 @ 5%; \$2500 @ 6%. **8.** \$5000 @ 4%; \$3000 @ 5%. **9.** \$5000 @ 4%; \$4000 @ 5%. **11.** \$9.00 per day. **12.** \$4.00; \$8.00. **13.** \$8.00; \$16.00. **14.** 6 @ \$6; 4 @ \$8. **16.** \$39. **17.** \$517.50. **18.** 6%. **19.** 2 yrs. **21.** \$600; \$90. **22.** \$1000. **24.** 12 nks.; 24 dms. **26.** 2 qtrs.; 4 dms.; 8 nks. **27.** 5 nks.; 5 dms.; 10 qtrs. **28.** 6 nks.; 4 qtrs.; 10 dms.

EXERCISE 21. *Page 79*

1. 30 lbs. **2.** 30. **3.** $58\frac{1}{3}$ oz. **4.** $\frac{23}{17}$ oz. **6.** $\frac{135}{16}$ oz. **7.** 28 lbs. **8.** 21 lbs. @ 15¢; 9 lbs. @ 25¢. **9.** 62.5 lbs. @ 22¢; 37.5 lbs. @ 30¢. **11.** 7 cases @ \$2.50; 8 cases @ \$3.

EXERCISE 22. *Page 83*

1. 4.8 ft. from 90 lb. boy. **2.** $93\frac{1}{3}$ lbs. **3.** 640 lbs. **4.** $333\frac{1}{3}$ lbs. **6.** $\frac{7}{5}$ ft. from fulcrum. **7.** $\frac{32}{9}$ ft. from end. **8.** 42 lbs. **9.** $\frac{56}{3}$ lbs. **11.** 500 lbs. **12.** $396\frac{2}{3}$ lbs. **13.** 550 lbs. **14.** 28 lbs.; 42 lbs. **16.** 10 ft. **17.** 3 ft.

EXERCISE 23. *Page 85*

1. $6\frac{6}{19}$ days. **2.** 75 min. **3.** A: 48 days; B: 192 days; C: 576 days. **4.** 4 min. **6.** 20 hrs. **7.** $9\frac{3}{4}$ min. **8.** $\dfrac{ab}{a+b}$ days. **9.** A: $\dfrac{5a+5}{a}$ days; B: $(5a+5)$ days.

EXERCISE 24. *Page 89*

1. 3. **2.** -2. **3.** -3. **4.** 12. **6.** $2a^2 - 3a - 2$. **7.** $\frac{1}{2}(a^2 + 3a - 4)$. **8.** $\frac{1}{2}(a^2 - 3a - 4)$. **9.** $2a^2 + 4ab + 2b^2 + 3a + 3b - 2$. **11.** 5. **12.** 4. **13.** 15. **14.** -6. **15.** $-\frac{14}{3}$. **16.** 1. **17.** 25. **18.** $-\frac{6}{5}$. **19.** $-\frac{5}{2}$. **21.** $-\frac{3}{2}$. **22.** $-\frac{1}{2}$. **23.** $\frac{1}{10}$. **24.** $-\frac{151}{60}$. **26.** $A = \pi r^2$ sq. units. **27.** $A = 5h$ sq. in. **28.** $A = 3b$ sq. ft. **29.** $A = s^2$ sq. units. **31.** $N = 88t$.

EXERCISE 26. *Page 96*

19. $y = x$; $x = y$. **21.** $y = 1 - x$; $x = 1 - y$. **22.** $y = x + 1$; $x = y - 1$. **23.** $y = 2x - 2$; $x = \frac{1}{2}(y + 2)$. **24.** $y = 6 - 3x$; $x = \dfrac{6 - y}{3}$. **26.** $y = \dfrac{2x + 6}{3}$; $x = \dfrac{3y - 6}{2}$. **27.** $y = \dfrac{6 - 3x}{2}$; $x = \dfrac{6 - 2y}{3}$. **28.** $y = 2x + 4$; $x = \dfrac{y - 4}{2}$. **29.** $y = 4 - 2x$; $x = \dfrac{4 - y}{2}$.
31. (*a*) $F = \frac{9}{5}C + 32$; (*b*) $C = \frac{5}{9}(F - 32)$; (*c*) $5(F + 40) = 9C + 360$.
32. (*a*) $P = \dfrac{I}{rt}$; (*b*) $r = \dfrac{I}{Pt}$; (*c*) $t = \dfrac{I}{Pr}$.

EXERCISE 28. *Page 105*

1. (2, 3). **2.** (2, 3). **3.** (2, 3). **4.** (−3, 5). **6.** $(\frac{4}{5}, -\frac{6}{5})$. **7.** $(-\frac{3}{2}, -\frac{5}{2})$. **8.** (−3, −1). **9.** (−2, 6). **11.** $(\frac{1}{3}, \frac{1}{2})$. **12.** (2, −3). **13.** (1, −4). **14.** (3, 2). **16.** (2, −3). **17.** (2, 0). **18.** (−2, 2). **19.** (4, 3). **21.** $(\frac{1}{3}, 1)$. **22.** $(-\frac{11}{2}, 4)$. **23.** (−7, 5). **24.** $(\frac{2}{5}, \frac{3}{2})$. **26.** (3, 6). **27.** (−7, 3). **41.** $(\frac{11}{5}, \frac{2}{5})$. **42.** $(\frac{36}{5}, -\frac{8}{5})$. **43.** $(\frac{19}{2}, -\frac{1}{2})$. **44.** $(\frac{35}{13}, -\frac{55}{78})$. **46.** $(-\frac{9}{29}, -\frac{63}{58})$. **47.** $(-\frac{10}{7}, -\frac{33}{7})$. **48.** $(-\frac{25}{12}, -\frac{4}{3})$. **49.** $(b, 0)$. **51.** $\left(\dfrac{a + b}{a}, \dfrac{a - b}{b}\right)$. **52.** $\left(a + b, \dfrac{a^2 - b^2}{b}\right)$. **53.** $(\frac{4}{3}, \frac{7}{5})$. **54.** $\left(-\dfrac{2}{a}, \dfrac{6}{b}\right)$. No solutions (inconsistent): **55, 58, 59,** and **61.** More than one solution (dependent): **56, 57, 60, 62,** and **63.** **66.** (1, 3). **67.** $(\frac{2}{5}, -\frac{3}{5})$. **68.** $(\frac{1}{2}, \frac{1}{3})$. **69.** $(\frac{1}{4}, 2)$.

EXERCISE 29. *Page 108*

1. (1, 2, 3). **2.** (1, −1, −2). **3.** (3, 3, 4). **4.** (1, −1, 0). **6.** $(\frac{1}{5}, -3, -\frac{22}{5})$. **7.** $(\frac{16}{29}, \frac{19}{29}, \frac{21}{29})$. **8.** $(\frac{17}{20}, -\frac{1}{20}, -\frac{9}{20})$. **9.** $(\frac{10}{11}, -\frac{3}{11}, \frac{3}{11})$. **11.** $(\frac{7}{3}, -\frac{2}{3}, \frac{4}{3}, \frac{5}{6})$. **12.** $(\frac{3}{25}, \frac{18}{25}, \frac{4}{5}, \frac{12}{25})$. **13.** $(3, \frac{1}{2}, 2)$. **14.** (1, 3, 4). **16.** $(\frac{9}{4}, \frac{9}{2}, 18)$.

EXERCISE 30. *Page 112*

1. 5. **2.** 8. **3.** −16. **4.** 10. **6.** −1. **7.** 31. **8.** 18. **9.** 0. **11.** 13. **12.** −11. **13.** 5. **14.** 57. **16.** 43. **17.** −12. **18.** 0. **19.** 0. **21.** 0.

EXERCISE 32. *Page 117*

1. 2; 1. **2.** 3; 5. **3.** 7; 5. **4.** 8; 6. **6.** 4; 5. **8.** 3 mph.; 4 mph. **9.** 2 mph.; 3 mph. **11.** 335 mph.; 15 mph. **12.** 150 mph.; 50 mph. **14.** 25. **16.** 347. **17.** 397. **18.** 9 in. × 5 in. **19.** 12 in. × 7 in.

21. 10 ft. × 4 ft. **22.** 16 in. × 9 in. **23.** \$2000 @ 5%; \$4000 @ 6%. **24.** \$6000 @ 5%; \$4000 @ 6%. **26.** \$3000 @ 4%; \$4000 @ 6%. **27.** A: \$10; B: \$6. **28.** 10 dms.; 20 nks. **29.** 6 qtrs.; 10 dms.; 20 nks. **31.** 70 lbs. 5%; 30 lbs. 15%. **32.** 4 lbs. @ $\frac{25}{2}$¢; 6 lbs. @ $\frac{25}{3}$¢. **33.** $\frac{200}{3}$ lbs. @ 20¢; $\frac{100}{3}$ lbs. @ 35¢. **34.** 20 lbs. meal; 30 lbs. flour.

EXERCISE 33. *Page 124*

26. $(ab^2c^3)^x$. **29.** $(2a)^4$. **44.** $\left(\frac{x}{y}\right)^2$.

EXERCISE 34. *Page 127*

1. (a) 1; -1. (b) 2; -2. (c) $\frac{1}{2}$; $-\frac{1}{2}$. (d) $\frac{2}{3}$; $-\frac{2}{3}$. (e) $\frac{5}{2}$; $-\frac{5}{2}$. (f) x, $-x$. **2.** (a) 3. (c) 2. (d) -3. (g) 4. (h) -1. (i) 1. (k) -2. **3.** (a) $\sqrt{5}$ in. (b) $\sqrt{13}$ in. (c) 5 in. (d) $\sqrt{58}$ in. (e) 13 in. (f) $2\sqrt{13}$ in. (g) 17 in. **4.** In cases (c), (e), and (g).

EXERCISE 35. *Page 130*

1. 2. **2.** 2. **3.** 27. **4.** 32. **6.** 125. **7.** -6. **8.** 16. **9.** 25. **11.** -9. **12.** -16. **13.** 4. **14.** -4. **16.** 3. **17.** 16. **18.** 8. **19.** -8. **21.** -4. **22.** 9. **23.** 9. **24.** -27. **26.** 4. **27.** 1. **28.** $\frac{1}{2}$. **29.** 3. **31.** $\frac{1}{4}$. **32.** 2. **33.** $\frac{1}{12}$. **34.** $\frac{3}{4}$. **36.** $-\frac{1}{27}$. **37.** $\frac{9}{4}$. **38.** $\frac{3}{2}$. **39.** $\frac{1}{81}$. **41.** $\frac{1}{3}$. **42.** $-.3$. **43.** -16. **44.** $-\frac{1}{5}$. **46.** $\frac{1}{3}$. **47.** $\frac{8}{17}$. **48.** $\frac{1}{y}$. **49.** $\frac{3}{2x^3}$. **51.** $\frac{2y^2}{x^3}$. **52.** $-\frac{1}{2x}$. **53.** $\frac{6y}{x^2}$. **54.** $\frac{y^2}{2}$. **56.** $\frac{2y^2}{x^2}$. **57.** $\frac{5y^4}{x^2}$. **58.** $\frac{xy^2}{4}$. **59.** $\frac{x}{y}$. **61.** $\frac{25y^8}{9x^8}$. **62.** $\frac{x^8}{y^{12}}$. **63.** $\frac{1}{18x^5y}$. **64.** $\frac{x^7y^3}{16}$. **66.** $\frac{xy^3}{12}$. **67.** $\frac{125y^4}{3x^8}$. **68.** $\frac{x^5}{2}$. **69.** $\frac{729x}{2y}$. **71.** $\frac{3a^4}{b^3}$. **72.** $\frac{(x-y)^2}{x+y}$. **73.** $\frac{a-b}{(a+b)^3}$. **74.** $\frac{x+2y}{2x-y}$. **76.** $\frac{1}{(2x-3y)^2}$. **77.** $\frac{1}{x-y}$. **78.** 1. **79.** 1. **81.** 1. **82.** $\frac{1}{(a^2+x^2)^{\frac{1}{4}}}$. **83.** $(x^2-a^2)^{\frac{1}{4}}$. **84.** $\frac{2a^2b^2}{b^2+a^2}$. **86.** $\frac{x^2y^2}{y^2-x^2}$. **87.** $\frac{ab(b^2+2ab-a^2)}{(a-b)^2(a+b)}$. **88.** $\frac{1}{x^2(a^2+x^2)^2}$. **91.** $\frac{a^2-x^2+1}{(a^2-x^2)^{\frac{3}{2}}}$. **92.** $\frac{a^2+x^2+1}{a^2+x^2}$. **93.** $\frac{a^2-x^2-1}{a^2-x^2}$. **94.** $\frac{a^2+x^2+1}{(a^2+x^2)^{\frac{3}{2}}}$. **96.** $\frac{a^2}{(a^2+x^2)^2}$. **97.** $\frac{a^2-x^2}{1-a^2}$. **98.** $\frac{a^2+x^2}{x^2}$. **99.** $\frac{(a^2-x^2)^{\frac{3}{2}}}{x^2-1}$. **101.** $\frac{(a^2-x^2)^{\frac{3}{2}}}{2a^2-x^2}$. **102.** $-\frac{(a^2+x^2)^2}{x^2}$.

EXERCISE 36. *Page 134*

1. 3. **2.** 4. **3.** 6. **4.** 2. **6.** 5. **7.** $2\sqrt{3}$. **8.** $2\sqrt{5}$. **9.** $2\sqrt{7}$. **11.** $3\sqrt{2}$. **12.** $3\sqrt{3}$. **13.** $3\sqrt{5}$. **14.** $3\sqrt{6}$. **16.** $6\sqrt{2}$. **17.** $3\sqrt{10}$. **18.** $-3\sqrt[3]{3}$. **19.** $-4\sqrt[3]{2}$. **21.** $4\sqrt{6}$. **22.** $8x\sqrt{2x}$. **23.** $5x\sqrt{5}$. **24.** $5x\sqrt{2x}$. **26.** $5\sqrt{6x}$. **27.** $5x^2\sqrt{7}$. **28.** $10x^2\sqrt{2x}$. **29.** $5\sqrt{10x}$. **31.** $6x^3\sqrt{2}$. **32.** $6x\sqrt[6]{5x}$. **33.** $6x^8\sqrt{6}$. **34.** $7x^4\sqrt{2}$. **36.** $7x^3\sqrt{5x}$. **37.** $2x\sqrt[3]{2x}$. **38.** $2x\sqrt[3]{3}$. **39.** $2x\sqrt[3]{4x}$. **41.** $2\sqrt[3]{6x^2}$. **42.** $2y\sqrt[3]{7}$. **43.** $2y^2\sqrt[3]{9y}$. **44.** $3b^2\sqrt[3]{2}$. **46.** $3y\sqrt[3]{3y}$. **47.** $3y^2\sqrt[3]{4y}$. **48.** $5x\sqrt[3]{2x^2}$. **49.** $5y^3\sqrt[3]{3}$. **51.** $2y\sqrt[4]{3y^3}$. **52.** $2x\sqrt[5]{x}$. **53.** $2\sqrt[5]{2x^4}$. **54.** $2x\sqrt[5]{3x}$. **56.** 9. **57.** $4\sqrt{10}$. **58.** 10. **59.** $9\sqrt{6}$. **61.** $4\sqrt[3]{2}$. **62.** 10. **63.** $6\sqrt[3]{4}$. **64.** 6. **66.** 10. **67.** 5. **68.** 14. **69.** $4x$. **71.** $3x\sqrt[3]{x}$. **72.** $3x\sqrt[3]{x^2}$. **73.** $2\sqrt{1-2x^2}$. **74.** $2\sqrt{1+3x^2}$. **76.** $3\sqrt{1+3x^2}$. **77.** $2\sqrt{1-4x^2}$. **78.** $2\sqrt{4-a^2}$. **79.** $5\sqrt{1+4x^2}$. **81.** $3\sqrt{x^2+4y^2}$. **82.** $5\sqrt{y^2+4x^2}$. **83.** 1. **84.** $\frac{2}{xy}\sqrt{y^2-x^2}$. **86.** $\sqrt{3}$. **87.** $\sqrt[3]{4}$. **88.** 2. **89.** 2. **91.** $\sqrt[3]{6}$. **92.** $\sqrt{5}$. **93.** $\sqrt{3}$. **94.** $3\sqrt{3}$. **96.** $\sqrt{2}$. **97.** $9\sqrt{3}$. **98.** $(3a+2b)\sqrt{3a}$. **99.** $3+\sqrt{2}$. **101.** $-1-\sqrt{2}$. **102.** $13\sqrt{2}$. **103.** $1-\sqrt{2}$. **104.** $2\sqrt{5x}$. **106.** $2-\sqrt{3}$. **107.** $\sqrt{5}-\sqrt{2}$. **108.** $\sqrt{3}-\sqrt{5}$. **109.** $-\sqrt{7}+\sqrt{2}$. **111.** $-3\sqrt{2}$. **112.** -12. **113.** $2x+50+20\sqrt{x}$. **114.** 59. **116.** $-4+2\sqrt{6}$. **117.** $10\sqrt{3}-18\sqrt{2}+6\sqrt{5}+3\sqrt{6}+10\sqrt{10}-60$. **121.** $x=\frac{a}{3b}$ or $x=-2$. **122.** $x=-1+\sqrt{2}$ or $x=-1-\sqrt{2}$. **123.** $x=\frac{c+\sqrt{m+p}}{4}$ or $x=\frac{c-\sqrt{m+p}}{4}$.

EXERCISE 37. *Page 138*

1. $\frac{\sqrt{2}}{2}=0.707$. **2.** $\frac{\sqrt{3}}{3}=0.577$. **3.** $\frac{\sqrt{6}}{2}=1.224$. **4.** $\frac{\sqrt{3}}{2}=0.866$. **6.** $\frac{\sqrt{15}}{5}=0.775$. **7.** $\frac{2\sqrt{5}}{5}=0.894$. **8.** $\frac{\sqrt{30}}{6}=0.913$. **9.** $\frac{\sqrt{6}}{4}=0.612$. **11.** $\frac{\sqrt{6}}{2}=1.224$. **12.** $\frac{\sqrt{15}}{3}=1.291$. **13.** $\frac{\sqrt{70}}{5}=1.661$. **14.** $\frac{\sqrt{115}}{5}=\frac{\sqrt{5}\sqrt{23}}{5}=2.145$. **16.** $\frac{\sqrt{45y^2-6xy}}{3y}$. **17.** $\frac{\sqrt{75+10y}}{5}$. **18.** $\frac{\sqrt{25x^2y^2-15}}{5xy}$. **19.** $\frac{\sqrt[3]{4}}{2}=0.794$. **21.** $\frac{\sqrt[3]{18}}{3}=0.874$. **22.** $\frac{\sqrt[3]{6}}{2}=0.908$. **23.** $\frac{\sqrt{x^2+x}}{x}$. **24.** $\frac{\sqrt{2x^2-3x}}{x}$. **26.** $\frac{\sqrt{x^3-2x}}{x}$. **27.** $\frac{\sqrt{x^3-x}}{x}$.

28. $\dfrac{\sqrt{3x^3 - x}}{x}$. **29.** $\dfrac{\sqrt{3x - x^3}}{x}$. **31.** $\dfrac{\sqrt{x - x^4}}{x}$. **32.** $\dfrac{\sqrt{x^5 - 2x}}{x^2}$.
33. $\dfrac{\sqrt{x^6 + x}}{x^2}$. **34.** $\dfrac{\sqrt{xz}}{z^2}$. **36.** $\dfrac{\sqrt{3x}}{3x^2}$. **37.** $\dfrac{3\sqrt{xz}}{z^2}$. **38.** $\dfrac{2y\sqrt{xy}}{x^2}$.
39. $\dfrac{\sqrt{xy}}{3x^2}$. **41.** $\dfrac{\sqrt{2} + 2}{2}$. **42.** $\dfrac{\sqrt{3} - 3}{3}$. **43.** $\dfrac{2\sqrt{5} + 5}{5}$. **44.** $\dfrac{3\sqrt{2} - 2}{2}$.
46. $5 - 2\sqrt{6}$. **47.** $5 + 2\sqrt{6}$. **48.** $\sqrt{3} - 1$. **49.** $-3 - 3\sqrt{2}$.
51. $\dfrac{3 - \sqrt{3}}{2}$. **52.** $\dfrac{5(\sqrt{3} + 1)}{2}$. **53.** $\dfrac{\sqrt{3x}}{3x}$. **54.** $\dfrac{\sqrt{3ab}}{ab}$.
56. $\dfrac{2(\sqrt{a} - \sqrt{b})}{a - b}$. **57.** $\dfrac{\sqrt[3]{9x}}{3x}$. **58.** $\dfrac{\sqrt[4]{2x^3}}{2x}$. **59.** $\dfrac{\sqrt[3]{25x^2}}{x}$.
61. $\dfrac{y^2\sqrt[3]{x^2y}}{x^2}$. **62.** $\dfrac{x + y + 2\sqrt{xy}}{x - y}$. **63.** $\dfrac{(\sqrt{x} - \sqrt{y})\sqrt{x - y}}{x - y}$. **64.** $\dfrac{b + a}{b - a}$.
66. $\dfrac{3\sqrt{2a}}{2}$. **67.** $\dfrac{\sqrt{a - b}}{a - b}$. **68.** $\dfrac{5\sqrt{6} - 5\sqrt{3} - \sqrt{2} + 2}{3}$. **69.** $\dfrac{\sqrt{6}}{2}$.
71. $\dfrac{1}{2 - \sqrt{2}}$. **72.** $\dfrac{-2}{3 + \sqrt{3}}$. **73.** $\dfrac{1}{5 - 2\sqrt{5}}$. **74.** $\dfrac{7}{2 + 3\sqrt{2}}$.
76. $\dfrac{1}{5 + 2\sqrt{6}}$. **77.** $\dfrac{1}{5 - 2\sqrt{6}}$. **78.** $\dfrac{y^3}{x\sqrt[3]{xy^2}}$. **79.** $\dfrac{x - y}{x + y - 2\sqrt{xy}}$.
81. $\dfrac{b + a}{b - a}$. **82.** $\dfrac{3x - 1}{(\sqrt{3x} - 1)\sqrt{3x + 1}}$. **83.** $\dfrac{3a}{\sqrt{2a}}$. **84.** $\dfrac{1}{\sqrt{a - b}}$.
86. $\dfrac{\sqrt{2}}{4}$. **87.** $\dfrac{\sqrt{5}}{25}$. **88.** $\dfrac{3\sqrt{6}}{4}$. **89.** $\dfrac{\sqrt[3]{4}}{4}$. **91.** $\dfrac{3\sqrt[3]{12}}{4}$.
92. $\dfrac{3\sqrt[3]{75}}{25}$. **93.** $\dfrac{2\sqrt{6}}{9}$. **94.** $\dfrac{\sqrt{2}}{4}$. **96.** $\dfrac{2\sqrt{10}}{25}$. **97.** $\dfrac{3\sqrt{6}}{4}$. **98.** $\dfrac{\sqrt[3]{4}}{4}$.
99. $\dfrac{2\sqrt[3]{18}}{9}$. **101.** $\dfrac{3\sqrt[3]{12}}{4}$.

EXERCISE 38. *Page 141*

1. $\sqrt[4]{9}$. **2.** $\sqrt[6]{8}$. **3.** $\sqrt[6]{4}$. **4.** $\sqrt[6]{9}$. **6.** $\sqrt{2}$. **7.** $\sqrt[10]{4}$. **8.** $\sqrt{2}$. **9.** $\sqrt[3]{3}$.
11. $\sqrt{3} > \sqrt[3]{5}$. **12.** $\sqrt{5} > \sqrt[3]{11}$. **13.** $\sqrt[4]{3} > \sqrt[3]{2}$. **14.** $\sqrt[6]{5} > \sqrt[3]{2}$.
16. $\sqrt[6]{32}$. **17.** $\sqrt[6]{243}$. **18.** $\sqrt[6]{108}$. **19.** $\sqrt[4]{8}$. **21.** $\sqrt{6}$. **22.** 1.
23. $\dfrac{\sqrt[4]{8}}{2}$. **24.** $\dfrac{\sqrt[3]{18}}{3}$. **26.** 1. **27.** $\sqrt[6]{3}$. **28.** $\dfrac{\sqrt[6]{x^5}}{x}$. **29.** $\sqrt[6]{x}$. **31.** $\sqrt[6]{x}$.
32. $\dfrac{x\sqrt{4 - x^2}}{2}$. **33.** $\dfrac{a\sqrt{3a}}{3}$. **34.** $\dfrac{x\sqrt[3]{2a^2x}}{2a}$. **36.** $\dfrac{y\sqrt[3]{12x}}{2}$. **37.** $\dfrac{\sqrt[4]{6yz}}{3yz}$.

EXERCISE 39. *Page 145*

4. $6i$. **6.** $3i$. **7.** $7i$. **8.** $i\sqrt{47}$. **9.** $8i$. **11.** $12i$. **12.** $-2i$.
13. $-3i$. **14.** $-i\sqrt{11}$. **16.** $3ia$. **17.** $4ix$. **18.** $-5iy^2$. **19.** $x^2i\sqrt{17}$.

l. $-ai\sqrt{23}$. **22.** -1. **23.** $-i$. **24.** -1. **26.** i. **27.** $2i$. **28.** $-2i$.
). $3 + 4i$. **31.** $-3 - 4i$. **32.** 13. **33.** 41. **34.** $9 + 7i$. **36.** 61.
'. $-4 + 3i$. **38.** $3 + 9i$. **39.** $16 - 12i$. **41.** $\frac{3+i}{5}$. **42.** $\frac{12-3i}{17}$.
l. $\frac{6-4i}{13}$. **44.** $\frac{15+10i}{13}$. **46.** $-i$. **47.** $\frac{3+11i}{10}$. **48.** $\frac{3-11i}{10}$.
). $\frac{8+i}{5}$. **51.** i. **52.** $\frac{-3i}{2}$. **53.** $\frac{2i}{3}$. **54.** i. **56.** $\frac{7i}{3}$. **57.** $\frac{10i}{3}$.
l. $-3i$. **59.** $\frac{5i}{3}$. **61.** -6. **62.** $-10 + 10i$. **63.** $6i$. **64.** $5i\sqrt{3}$.
). $\sqrt{2}$. **67.** $i\sqrt{2}$. **68.** $\frac{10-15i}{13}$. **69.** $\frac{1-i}{6}$. **71.** $-1 - i$.

XERCISE 40. *Page 148*

3. (*a*) Unlimited number. (*b*) No. (*c*) No.

XERCISE 41. *Page 152*

(The values of a, b, and c are listed in that order.) **2.** 4, -4, 1.
2, 1, -6. **4.** 3, 2, -1. **6.** 1, 2, -3. **7.** 4, 5, -6. **8.** 2, 1, -3.
1, 3, 2. **11.** 4, 4, -3. **12.** 4, 8, 3. **13.** 1, -2, 0. **14.** 1, -3, 0.
). 3, 2, 0. **17.** 2, 3, 0. **18.** 1, 0, $-d^2$. **19.** 4, 0, $-9e^4$. **21.** $4d^2$, 0, $-e^2$.
:. 1, $r - s$, $-rs$. **23.** a^2, 0, $-16b^2$. **26.** $2 - C$, $-A - B$, $A + 4$.
'. $A - 3$, $1 - B - C$, 1. **28.** $A - B$, $A + C$, $B - C$. **29.** $A - C$,
$\cdot D$, B.

XERCISE 42. *Page 155*

1. 2; -2. **2.** 2; -2. **3.** 2; -2. **4.** 3; -3. **6.** 5; -5. **7.** $3i$; $-3i$.
$2i$; $-2i$. **9.** $2i$; $-2i$. **11.** $7i$; $-7i$. **12.** $5i$; $-5i$. **13.** $3\sqrt{2}$; $-3\sqrt{2}$.
l. $2\sqrt{3}$; $-2\sqrt{3}$. **16.** $\frac{4\sqrt{2}}{a}$; $\frac{-4\sqrt{2}}{a}$. **17.** $\frac{2\sqrt{3}}{b}$; $\frac{-2\sqrt{3}}{b}$. **18.** $\frac{4c\sqrt{6}}{a}$;
$\frac{\cdot 4c\sqrt{6}}{a}$. **19.** $3ai\sqrt{2}$; $-3ai\sqrt{2}$. **21.** $\frac{10bi}{a}$; $\frac{-10bi}{a}$. **22.** $4bi\sqrt{2}$; $-4bi\sqrt{2}$.
l. $2ci\sqrt{6}$; $-2ci\sqrt{6}$. **24.** $\frac{4ib\sqrt{6}}{a}$; $\frac{-4ib\sqrt{6}}{a}$. **25.** $\frac{5i\sqrt{ab}}{a}$; $\frac{-5i\sqrt{ab}}{a}$.
). $\frac{10i\sqrt{c}}{a}$; $\frac{-10i\sqrt{c}}{a}$. **27.** $\frac{2i\sqrt{5ab}}{a}$; $\frac{-2i\sqrt{5ab}}{a}$. **28.** -4; 3. **29.** 4; -3.
l. 4; -1. **32.** 3; -2. **33.** 5; -2. **34.** $\frac{1}{2}$; -1. **36.** $-\frac{1}{3}$; 1.
'. 1; $-\frac{1}{4}$. **38.** $\frac{1}{2}$; $\frac{1}{2}$. **39.** $-\frac{1}{3}$; $-\frac{1}{3}$. **41.** $4a$; $4a$. **42.** $-\frac{5}{3}$; $-\frac{5}{3}$.
l. 0; $\frac{1}{6}$. **44.** 0; $\frac{1}{3}$. **46.** 0; $-\frac{a}{10}$. **47.** 0; $-\frac{b}{a}$. **48.** 0; $\frac{d}{c}$. **49.** $\frac{1}{2}$; $-\frac{2}{3}$.
l. $\frac{3}{2}$; $-\frac{1}{5}$. **52.** $-\frac{2}{5}$; $\frac{1}{2}$. **53.** $\frac{1}{3}$; $-\frac{2}{5}$. **54.** $\frac{1}{3}$; $-\frac{2}{5}$. **56.** $-\frac{2}{5}$; $-\frac{1}{2}$.

57. $\frac{2}{5}; \frac{1}{2}$. **58.** $-\frac{2}{3}; -\frac{3}{7}$. **59.** $\frac{2}{3}; \frac{3}{7}$. **61.** $4; -\frac{5}{3}$. **62.** $\frac{\sqrt{39}}{6}; -\frac{\sqrt{39}}{6}$.
63. $20; -8$. **64.** $\frac{i\sqrt{15}}{3}; -\frac{i\sqrt{15}}{3}$. **66.** $\frac{\sqrt{6}}{4}; -\frac{\sqrt{6}}{4}$. **67.** $\frac{1}{7}$. **68.** $0; \frac{7}{10}$.
69. $\frac{11}{5}$.

EXERCISE 43. *Page 159*

1. $4; 2$. **2.** $-3; -2$. **3.** $-4; 2$. **4.** $5; -2$. **6.** $-5; 2$. **7.** $-\frac{1}{3}; \frac{3}{2}$.
8. $-\frac{2}{3}; \frac{1}{2}$. **9.** $\frac{1}{3}; -\frac{3}{2}$. **11.** $\frac{3}{4}; -\frac{4}{3}$. **12.** $\frac{3}{2}; -\frac{3}{4}$. **13.** $\frac{1 \pm \sqrt{7}}{3}$.
14. $\frac{-3 \pm \sqrt{69}}{10}$. **16.** $\frac{-2 \pm \sqrt{19}}{3}$. **17.** $\frac{-3 \pm \sqrt{19}}{2}$. **18.** $\frac{-1 \pm \sqrt{7}}{4}$.
19. $\frac{7 \pm 2\sqrt{7}}{7}$. **21.** $\frac{-3 \pm i\sqrt{71}}{10}$. **22.** $\frac{-5 \pm i\sqrt{23}}{6}$. **23.** $\frac{-3 \pm i\sqrt{3}}{4}$.
24. $\frac{-3 \pm i\sqrt{71}}{20}$. **26.** $1 \pm 2i$. **27.** $\frac{3 \pm i\sqrt{15}}{2}$. **28.** $2 \pm i\sqrt{2}$.
29. $3 \pm \sqrt{7}$. **31.** $\frac{1 \pm \sqrt{21}}{2}$. **32.** $\frac{-1 \pm i\sqrt{3}}{2}$. **33.** $\frac{1 \pm \sqrt{53}}{2}$.
44. $\frac{3 \pm \sqrt{201}}{6}$.

EXERCISE 44. *Page 163*

1. $\frac{3 \pm i\sqrt{31}}{4}$. **2.** $\frac{-5 \pm \sqrt{41}}{4}$. **3.** $\frac{1}{2}; \frac{1}{2}$. **4.** $\frac{3}{5}; -1$. **6.** $3 \pm 3\sqrt{2}$.
7. $\frac{-1 \pm \sqrt{13}}{2}$. **8.** $\frac{-3 \pm \sqrt{13}}{2}$. **9.** $\frac{3 \pm \sqrt{5}}{2}$. **11.** $\frac{1 \pm i\sqrt{11}}{2}$. **12.** $\frac{5}{3}; \frac{5}{3}$.
13. $\frac{-1 \pm i\sqrt{3}}{2}$. **14.** $\frac{5 \pm \sqrt{17}}{4}$. **16.** $\frac{-1 \pm \sqrt{17}}{8}$. **17.** $5 \pm 2\sqrt{6}$.
18. $\frac{1 \pm i\sqrt{39}}{20}$. **19.** $\frac{5 \pm \sqrt{13}}{6}$. **21.** $1; \frac{1}{4}$. **22.** $1; \frac{1}{7}$. **23.** $\frac{1 \pm i\sqrt{7}}{6}$.
24. $\frac{1}{2}; \frac{1}{3}$. **26.** $1; -\frac{1}{8}$. **27.** $\frac{9 \pm \sqrt{57}}{6}$. **28.** $1; -\frac{1}{5}$. **29.** $\frac{1}{2}; -\frac{1}{5}$.
31. $\frac{3 \pm \sqrt{65}}{14}$. **32.** $\frac{1}{2}; \frac{1}{6}$. **33.** $\frac{3 \pm \sqrt{7}}{4}$. **34.** $1; \frac{1}{11}$. **36.** $\frac{1}{2}; \frac{1}{3}$.
37. $\frac{b \pm \sqrt{b^2 + 4ac}}{2a}$. **38.** $\frac{-c \pm \sqrt{c^2 - 4ab}}{2b}$. **39.** $\frac{-b^2 \pm \sqrt{b^4 - 4a^2c^2}}{2a^2}$.
41. $\frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$. **42.** $\frac{b \pm \sqrt{b^2 + 16ac}}{4a}$. **43.** $\frac{3 \pm \sqrt{5}}{2}$.
44. $\frac{-1 \pm \sqrt{17}}{2}$. **46.** $\frac{v_0 \pm \sqrt{v_0^2 - 2ag}}{g}; \frac{2a + gt^2}{2t}$.

47. $\dfrac{-2a + d \pm \sqrt{4a^2 - 4ad + d^2 + 8ds}}{2d}$. **48.** $-1 \pm \sqrt{2 - x^2 - 4x}$.

49. $\dfrac{1 \pm \sqrt{1 - 4g^2x^2}}{2g}$; $\dfrac{\pm \sqrt{gR - g^2R^2}}{g}$.

EXERCISE 46. *Page 169*

1. $x^2 - 3x + 2 = 0$. **2.** $x^2 - 2x - 3 = 0$. **3.** $x^2 + 4x + 3 = 0$. **4.** $x^2 - 2x = 0$. **6.** $8x^2 - 10x + 3 = 0$. **7.** $6x^2 - x - 2 = 0$. **8.** $5x^2 - 2x = 0$. **9.** $5x^2 - 2x - 3 = 0$. **11.** $x^2 - 6x + 7 = 0$. **12.** $2x^2 - 4x - 1 = 0$. **13.** $2x^2 - 2x - 1 = 0$. **14.** $x^2 + 1 = 0$. **16.** $x^2 - 4x + 13 = 0$. **17.** $x^2 - 4x + 7 = 0$. **18.** $4x^2 - 8x + 5 = 0$.

EXERCISE 47. *Page 171*

1. $-8, -5; 5, 8$. **2.** 12. **3.** 150 ft. $\times$ 200 ft. **4.** 50 ft. $\times$ 120 ft. **6.** 8 in. $\times$ 5 in. **7.** 6 in. $\times$ 3 in. **8.** $2 + 2\sqrt{2}$ ft. **9.** 15 in. $\times$ 8 in. **11.** $(8 + 2\sqrt{17})$ in. **12.** $(4 + 2\sqrt{5})$ in. **13.** $-\frac{1}{2}$. **14.** $-\frac{[illegible]}{3}$. **16.** $8; -3$. **17.** $\pm 2\sqrt{6}$.

EXERCISE 48. *Page 176*

1. $-1; \dfrac{1 \pm i\sqrt{3}}{2}$. **2.** $\pm 1; \pm i$. **3.** $i; i; -i; -i$. **4.** $0; 1; 1; -1; -1$. **6.** $0; 0; 3; -2$. **7.** $1; 1; -1$. **8.** $0; 1; i; -i$. **9.** $2; \dfrac{\pm i\sqrt{6}}{2}$.

11. $\sqrt{2}; -\sqrt{2}; i\sqrt{2}; -i\sqrt{2}$. **12.** $\dfrac{1 \pm i\sqrt{3}}{2}; \dfrac{-1 \pm i\sqrt{3}}{2}$.

13. $x^3 - 6x^2 + 11x - 6 = 0$. **14.** $x^3 + 4x^2 + x - 6 = 0$. **16.** $x^3 + x^2 - 6x = 0$. **17.** $x^4 - 1 = 0$. **18.** $x^3 - 2x^2 - x = 0$. **19.** $x^3 - 2x^2 + 3x = 0$. **21.** $8x^3 - 20x^2 + 18x - 5 = 0$. **22.** $4x^4 - 16x^3 + 13x^2 + 6x - 1 = 0$. **23.** $6x^4 - 7x^3 - x^2 + 2x = 0$. **24.** $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$. **26.** $x^3 = 0$. **27.** $x^3 + 3x^2 + 3x + 1 = 0$. **28.** $x^4 - 2x^3 + x^2 = 0$.

EXERCISE 49. *Page 178*

1. $\pm\sqrt{2}; \dfrac{\pm i\sqrt{6}}{2}$. **2.** $\pm \dfrac{\sqrt{6}}{2}; \pm i\sqrt{2}$. **3.** $\pm \dfrac{\sqrt{3}}{3}; \pm \dfrac{i\sqrt{2}}{2}$.

4. $\pm \dfrac{i\sqrt{3}}{3}; \pm \dfrac{\sqrt{2}}{2}$. **6.** $\pm \dfrac{\sqrt{2}}{2}; \pm i$. **7.** $-2; 1 \pm i\sqrt{3}; 1; \dfrac{-1 \pm i\sqrt{3}}{2}$.

8. $2; -1 \pm i\sqrt{3}; -1; \dfrac{1 \pm i\sqrt{3}}{2}$. **9.** $-3; \dfrac{3 \pm 3i\sqrt{3}}{2}; 1; \dfrac{-1 \pm i\sqrt{3}}{2}$.

11. $2;\ -\frac{1}{2};\ -1 \pm i\sqrt{3};\ \frac{1 \pm i\sqrt{3}}{4}.$ **12.** $\frac{1}{2};\ -2;\ 1 \pm i\sqrt{3};\ \frac{-1 \pm i\sqrt{3}}{4}.$
13. $1;\ 3;\ \frac{-1 \pm i\sqrt{3}}{2};\ \frac{-3 \pm 3i\sqrt{3}}{2}.$ **14.** $-1;\ \frac{3}{2};\ \frac{-3 \pm 3i\sqrt{3}}{4};\ \frac{1 \pm i\sqrt{3}}{2}.$
16. $\frac{-1 \pm \sqrt{7}}{2};\ \frac{-1 \pm \sqrt{5}}{2}.$ **17.** $\frac{1 \pm \sqrt{7}}{2};\ \frac{1 \pm i\sqrt{3}}{2}.$ **18.** $\frac{1 \pm \sqrt{5}}{2};$ $\frac{1 \pm \sqrt{13}}{2}.$ **19.** $1;\ -2;\ \frac{-1 \pm \sqrt{13}}{2}.$ **21.** $1;\ -\frac{1}{2};\ \frac{1 \pm i\sqrt{15}}{4}.$
22. $0;\ -1;\ \frac{-1 \pm \sqrt{5}}{2}.$ **23.** $1;\ 1;\ \frac{1 \pm i\sqrt{15}}{4}.$ **24.** $1 \pm \sqrt{2};\ \frac{-1 \pm \sqrt{17}}{4}.$

EXERCISE 50. *Page 180*

1. 14. **2.** 20. **3.** 0. **4.** 1; 2. **6.** -1. **7.** No root. **8.** -2. **9.** $\frac{1}{2}$. **11.** 3; -3. **12.** 1. **13.** 2. **14.** 1. **16.** -1; -2. **17.** $-a$; $-b$. **18.** 1; -3.

EXERCISE 51. *Page 184*

1. $3x^2 + x + 5.$ **2.** $5x^3 - 2x^2 + x - 2.$ **3.** $2x^3 - 3x^2 + 2x - 3.$ **4.** $3x^3 - 2x + 1.$ **6.** $x^4 - 1.$ **7.** $3x^2 - 8.$ **8.** $2x^3 - 3.$
9. $3x^2 + 4x + 14 + \frac{46}{x - 3}.$ **11.** $3x^3 + 3x^2 + x + 1 + \frac{6}{x - 1}.$
12. $4x^4 - 4x^3 + x^2 + x - 1 - \frac{4}{x + 1}.$ **13.** $7x^2 - 34x + 141 - \frac{564}{x + 4}.$
14. $3x^3 - 2x.$ **16.** Yes. **17.** Yes. **18.** Yes. **19.** Yes. **21.** No.

EXERCISE 52. *Page 187*

1. ± 1; 3. **2.** 1; 1; -2. **3.** $1;\ \frac{-1 \pm i\sqrt{11}}{2}.$ **4.** 1. **6.** -1.
7. $\pm 1;\ \frac{1}{2}.$ **8.** $-1;\ -\frac{1}{6};\ -2.$ **9.** $\pm 2;\ \frac{-3 \pm \sqrt{5}}{2}.$ **11.** $-1;\ 3;\ \pm\sqrt{3}.$
12. $1;\ -\frac{1}{2};\ \pm\frac{\sqrt{5}}{2}.$ **13.** $\frac{1}{2};\ -1;\ \pm\frac{2\sqrt{3}}{3}.$ **14.** No rational roots.
16. $1;\ \frac{-1 \pm i\sqrt{7}}{2}.$ **17.** $-1;\ \pm 4;\ \frac{1}{3}.$ **18.** 1. **19.** $2;\ 2;\ -2;\ -1 \pm i\sqrt{3}.$
21. $1;\ -9;\ \pm\sqrt{3}.$ **22.** No rational roots.

EXERCISE 54. *Page 194*

1. (4, 3); $(-4, -3)$. **2.** (5, 0); (0, 5). **3.** $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right);$ $\left(\frac{-5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2}\right).$ **4.** $(4, 3);\ \left(\frac{-56}{13}, \frac{-33}{13}\right).$ **6.** (No real intersections.)

$(i\sqrt{11}, 6)$; $(-i\sqrt{11}, 6)$. **7.** $(5, 3)$; $(-5, -3)$. **8.** $(5, 3)$; $(-5, 3)$. **9.** $(4, 0)$. **11.** $(3, i\sqrt{7})$; $(3, -i\sqrt{7})$. **12.** $(2, 4)$; $(-\frac{2}{3}, \frac{4}{9})$. **13.** $(1, 1)$. **14.** $(1 + i, 2i)$; $(1 - i, -2i)$. **16.** $(4, 2)$; $(4, -2)$. **17.** $(1, 1)$. **18.** $(-1, \frac{1}{2})$; $(1, -\frac{1}{2})$. **19.** $(1, -\frac{3}{4})$; $(2, -\frac{1}{2})$. **21.** (a, b); $\left(\frac{b}{2}, 2a\right)$. **22.** $\left(\frac{a}{2}, -2b\right)$; $(b, -a)$. **23.** 10 rods × 2 rods. **24.** 12 in. × 5 in.

EXERCISE 55. *Page 197*

1. $\left(\frac{\sqrt{58}}{2}, \frac{\sqrt{42}}{2}\right)$; $\left(\frac{-\sqrt{58}}{2}, \frac{-\sqrt{42}}{2}\right)$; $\left(\frac{\sqrt{58}}{2}, \frac{-\sqrt{42}}{2}\right)$; $\left(\frac{-\sqrt{58}}{2}, \frac{\sqrt{42}}{2}\right)$. **2.** $(5, 0)$; $(-5, 0)$. **3.** $\left(\frac{\sqrt{10}}{2}, \frac{i\sqrt{6}}{2}\right)$; $\left(\frac{\sqrt{10}}{2}, \frac{-i\sqrt{6}}{2}\right)$; $\left(\frac{-\sqrt{10}}{2}, \frac{i\sqrt{6}}{2}\right)$; $\left(\frac{-\sqrt{10}}{2}, \frac{-i\sqrt{6}}{2}\right)$. **4.** $(0, 2)$; $(0, -2)$. **6.** $(1, 1)$; $(-1, 1)$. **7.** $(2, -1)$; $(-2, 1)$. **8.** $\left(-\frac{i\sqrt{2}}{2}, i\sqrt{2}\right)$; $\left(\frac{i\sqrt{2}}{2}, -i\sqrt{2}\right)$. **9.** $\left(\frac{\sqrt{3}}{2}, \frac{2}{3}\right)$; $\left(-\frac{\sqrt{3}}{2}, \frac{2}{3}\right)$. **11.** $(\frac{2}{3}, \frac{3}{2})$. **12.** $(3, 1)$. **13.** $\left(i, -\frac{i}{2}\right)$; $\left(-i, \frac{i}{2}\right)$. **14.** $\left(-\frac{i}{6}, 2i\right)$; $\left(\frac{i}{6}, -2i\right)$. **16.** $\left(\sqrt{a}, \frac{b\sqrt{a}}{a}\right)$; $\left(-\sqrt{a}, -\frac{b\sqrt{a}}{a}\right)$.

EXERCISE 56. *Page 200*

1. $(3, 4)$; $(3, -4)$; $(3, -4)$; $(-4, 3)$. **2.** $(0, 0)$; $(4, 4)$; $(-1, \frac{1}{4})$; $(-1, \frac{1}{4})$. **3.** $(5, 3)$; $(-5, 3)$; $(5, 3)$; $(-5, -3)$. **4.** $(5, 3)$; $(5, 3)$; $(-\frac{13}{3}, -\frac{5}{3})$. **6.** $(\frac{17}{2}, \frac{15}{2})$; $(\frac{17}{2}, -\frac{15}{2})$. **7.** $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$; $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$; $(\frac{1}{2}, \frac{1}{2})$; $(-\frac{1}{2}, -\frac{1}{2})$. **8.** $(1, 2)$; $(-1, -2)$; $(1, 2)$; $(11, -8)$. **9.** $(1, 1)$; $(1, 1)$; $(\frac{2}{3}, \frac{1}{3})$; $(-\frac{5}{3}, \frac{11}{3})$. **11.** $(0, 0)$; $(0, 0)$; $(0, 0)$; $(0, 0)$. **12.** $(\frac{2}{5}, -\frac{6}{5})$; $(\frac{3}{2}, 1)$; $(-\frac{2}{5}, \frac{6}{5})$; $(-\frac{3}{2}, -1)$. **13.** $(-2, 6)$; $(4, -6)$; $(0, -2)$; $(-6, 10)$. **14.** $(\frac{1}{3}, \frac{2}{3})$; $(-\frac{1}{3}, -\frac{2}{3})$; $(\frac{3}{2}, -\frac{1}{2})$; $(-\frac{3}{2}, \frac{1}{2})$. **16.** $(1, 2)$; $(-1, -2)$; $(\frac{1}{2}\sqrt{10}, -\frac{1}{2}\sqrt{10})$; $(-\frac{1}{2}\sqrt{10}, \frac{1}{2}\sqrt{10})$. **17.** $(1, 1)$; $(-1, -1)$; $(1, 1)$; $(-1, -1)$. **18.** $\left(\frac{\sqrt{14}}{7}, \frac{2\sqrt{14}}{7}\right)$; $\left(-\frac{\sqrt{14}}{7}, -\frac{2\sqrt{14}}{7}\right)$.

EXERCISE 57. *Page 203*

1. $\frac{1}{2}$. **2.** $\frac{7}{15{,}840}$. **3.** $\frac{352}{5}$. **4.** $\frac{3}{32}$. **6.** $\frac{14{,}400}{13}$. **7.** x^2. **8.** -1. **9.** a. **11.** 2. **12.** -9; 1. **13.** $\frac{5}{7}$; $-\frac{1}{2}$. **14.** $-\frac{1}{20}$; $-\frac{2}{3}$. **16.** $(2, 2)$; $(-2, -2)$. **17.** $(\frac{1}{3}, 0)$. **18.** $(-4, 0)$. **19.** $(\frac{2}{3}, 2)$. **21.** $(\frac{5}{2}, -2, 2)$. **22.** $(2, 4, 6)$.

EXERCISE 58. *Page 205*

1. $\frac{a+c}{c} = \frac{b+d}{d}$; $\frac{a-c}{c} = \frac{b-d}{d}$; $\frac{a+c}{a-c} = \frac{b+d}{b-d}$. **2.** $\frac{b+a}{a} = \frac{d+c}{c}$; $\frac{b-a}{a} = \frac{d-c}{c}$; $\frac{b+a}{b-a} = \frac{d+c}{d-c}$. **3.** $\frac{b}{a+b} = \frac{d}{c+d}$; $\frac{b}{a-b} = \frac{d}{c-d}$; $\frac{a-b}{a+b} = \frac{c-d}{c+d}$; $\frac{a}{b+a} = \frac{c}{c+d}$; $\frac{a}{b-a} = \frac{c}{d-c}$; $\frac{b-a}{b+a} = \frac{d-c}{d+c}$; $\frac{c}{a+c} = \frac{d}{b+d}$; $\frac{c}{a-c} = \frac{b}{b-d}$; $\frac{a-c}{a+c} = \frac{b-d}{b+d}$. **16.** $18\frac{2}{3}$ ft. **17.** $12\frac{6}{7}$ ft. **18.** 24 ft. **19.** $\frac{15}{7}$ mis.

EXERCISE 59. *Page 213*

1. $y = kx^3$; $\frac{y}{x^3} = k$; x varies directly as $\sqrt[3]{y}$.

2. $z = kxy$; $\frac{z}{xy} = k$; x varies directly as z and inversely as y.

3. $w = \frac{kx^2y}{z}$; $\frac{zw}{x^2y} = k$; x varies directly as $\sqrt{wz}$ and inversely as $\sqrt{y}$.

4. $y = \frac{kw}{x^2z}$; $\frac{x^2yz}{w} = k$; x varies directly as $\sqrt{w}$ and inversely as $\sqrt{yz}$.

6. $w = \frac{kx}{yz^3}$; $\frac{wyz^3}{x} = k$; x varies directly as w, y, and z^3 (or as wyz^3).

7. $x^2 = \frac{ky}{z^2w}$; $\frac{x^2z^2w}{y} = k$; x varies directly as $\sqrt{y}$ and inversely as $z\sqrt{w}$.

8. $y^3 = \frac{kxz}{w}$; $\frac{y^3w}{xz} = k$; x varies directly as wy^3 and inversely as z.

9. (*a*) $\frac{2}{3}$; (*b*) $i\sqrt{3}$; (*c*) $\frac{48}{5}$; (*d*) 24. **11.** $\frac{45}{4}$ sq. in.; 20 sq. in.; $\frac{245}{4}$ sq. in.; 80 sq. in.; $\frac{405}{4}$ sq. in. **12.** 9 pts. **13.** 115.74 lbs.; 154.05 lbs. **14.** $2\sqrt[3]{2}$ ft.; $2\sqrt[3]{5}$ ft.; $2\sqrt[3]{10}$ ft.; $2\sqrt[3]{20}$ ft.; $2\sqrt[3]{60}$ ft.; $4\sqrt[3]{15}$ ft. **16.** Moon: 16.46 lbs.; Mercury: 26.3 lbs.; Venus: 86.1 lbs.; Mars: 43.1 lbs.; Jupiter: 252.8 lbs.; Saturn: 107.5 lbs.; Sun: 2777.5 lbs. **17.** 199.5 lbs. **18.** S is decreased by $\frac{S}{9}$. **19.** (*a*) It is decreased to $11\frac{31}{64}$ units; (*b*) $96\sqrt{2}$ in. **21.** 168 cu. ft. **22.** Mass of Jupiter = 302.5 (mass of earth).

EXERCISE 60. *Page 219*

1. $x^3 + 3x^2y + 3xy^2 + y^3$. **2.** $x^3 - 3x^2y + 3xy^2 - y^3$. **3.** $64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6$. **4.** $27a^6 + 27a^4b + 9a^2b^2 + b^3$. **6.** $16x^2 + 32xy\sqrt{x} + 24xy^2 + 8y^3\sqrt{x} + y^4$. **7.** $729x^6 - 729x^5y + \frac{1215}{4}x^4y^2 - \frac{135}{2}x^3y^3 + \frac{135}{16}x^2y^4 - \frac{9}{16}xy^5 + \frac{y^6}{64}$. **8.** $a^8 + \frac{4}{3}a^6b +$

$\frac{2}{3}a^4b^2 + \frac{4}{27}a^2b^3 + \frac{b^4}{81}$. **9.** $x^5 - 5x^4y^2 + 10x^3y^4 - 10x^2y^6 + 5xy^8 - y^{10}$.
11. $\frac{x^4}{16} + \frac{x^3y^2}{6} + \frac{x^2y^4}{6} + \frac{2xy^6}{27} + \frac{y^8}{81}$. **12.** $\frac{x}{8} - \frac{3y\sqrt[3]{x^2}}{4} + \frac{3y^2\sqrt[3]{x}}{2} - y^3$.
13. $x^5 - 20x^{\frac{9}{2}}y + 180x^4y^2 - 960x^{\frac{7}{2}}y^3 + \cdots$. **14.** $\frac{256}{x^8} - \frac{256}{x^6} + \frac{112}{x^4} - \frac{28}{x^2} + \cdots$. **16.** -70. **17.** $-792a^2b^{17}$. **18.** 3432. **19.** $x^{-3} - 3x^{-4}y + 6x^{-5}y^2 - 10x^{-6}y^3 + \cdots$. **21.** $\frac{1}{\sqrt{2x}}\left(1 - \frac{y}{4x} + \frac{3y^2}{32x^2} - \frac{5y^3}{128x^3} + \cdots\right)$.
22. $\frac{\sqrt{a}}{a} - \frac{2\sqrt{b}}{a} + \frac{4b\sqrt{a}}{a^2} - \frac{8b\sqrt{b}}{a^2} + \cdots$. **23.** $1 - 2x + 5x^2 - \frac{40x^3}{3} + \cdots$.
24. $1 + 2x + 4x^2 + 8x^3 + \cdots$. **26.** $a^2 + b^2 + c^2 + d^2 + 2ab - 2ac - 2ad - 2bc - 2bd + 2cd$. **27.** $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.
28. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$. **29.** $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

EXERCISE 61. *Page 224*

1. 11. **2.** -5. **3.** $\frac{11}{2}x$. **4.** -6. **6.** 7, 11. **7.** -4, -10, -16. **8.** 8, 6, 4, 2, 0. **9.** 21. **11.** 4. **12.** 115. **13.** 8. **14.** 21. **16.** 200 mis. **17.** 25 mis. **18.** $20.

EXERCISE 62. *Page 226*

1. 48. **2.** 185. **3.** -88. **4.** 114. **6.** 18; $\frac{19}{17}$. **7.** 192; 27. **8.** 147; 7. **9.** 31; -3.6. **11.** 312; 15. **12.** -60; -3. **13.** 11; 27. **14.** 7; 23. **16.** 3; 24. **17.** 21. **18.** 150 ft. **19.** 440 ft. **21.** $3000; $2800; $2600; $2400; $2200. **22.** $1275. **23.** $6840. **24.** $2950.

EXERCISE 63. *Page 231*

1. 32. **2.** 243. **3.** $\frac{1}{2}$. **4.** -128. **6.** 4, 8, 16, or -4, 8, -16. **7.** 16, 8, 4, 2. **8.** -8, 4. **9.** -3, 9, -27, 81. **11.** ± 4. **12.** ± 3. **13.** ± 9. **14.** ± 10. **16.** 4; 30. **17.** 6; 21. **18.** 5; 121. **19.** 6; -21. **21.** 6; 1. **22.** 5; 1. **23.** $\frac{1}{2}$; 255. **24.** $\frac{1}{2}$; 31.5. **26.** $10.23. **27.** $3000; $1500; $750; $375. **28.** $4000; $3200; $2560; $2048. **29.** 4096.

EXERCISE 64. *Page 236*

1. 1.08. **2.** 1.26. **3.** 1.44. **4.** 0.90. **6.** 1.5. **7.** -0.18. **8.** 0.18. **9.** 0.42. **11.** 0.70. **12.** 1.4. **13.** 2.1. **14.** -0.52.

16. -0.92. **17.** 25. **18.** $\frac{1}{4}$. **19.** 4. **21.** 125. **22.** $\frac{1}{3}$. **23.** 3. **24.** $\frac{1}{2}$. **26.** -1. **27.** $-\frac{2}{3}$. **28.** $\frac{3}{2}$. **29.** $2\sqrt{2}$. **31.** $\frac{1}{9}$. **32.** 512. **33.** $\frac{7}{4}$. **34.** $\frac{\sqrt[3]{4}}{4}$. **36.** Change left side to $\log_a \sqrt[3]{B}$. **37.** Change left side to $\log_a A^{\frac{2}{3}}$. **38.** Change right side to $10(\log_a x + \log_a y)$. **39.** True. **41.** Change right side to $\frac{1}{2}(\log_{10} x + \log_{10} y - \log_{10} z)$. **42.** Change right side to $\frac{1}{3}(2 \log_{10} A - \log_{10} B)$. **43.** Change left side to $\log_a \sqrt[5]{C}$. **44.** True. **46.** True. **47.** True.

EXERCISE 66. *Page 244*

1. 2.2625. **2.** 2.5403. **3.** 2.7210. **4.** 2.3711. **6.** 2.6232. **7.** 2.5563. **8.** 2.9657. **9.** 1.5119. **11.** 0.5705. **12.** 0.9180. **13.** 8.7924 − 10. **14.** 9.8663 − 10. **16.** 6.9590 − 10. **17.** 6.7782 − 10. **18.** 7.7664 − 10. **19.** 6.8069 − 10. **21.** 11.5. **22.** 135. **23.** 22.6. **24.** 430. **26.** 3920. **27.** 0.289. **28.** 0.0315. **29.** 0.00231. **31.** 5.48. **32.** 1.01. **33.** 0.000100.

EXERCISE 67. *Page 246*

1. 3.4041. **2.** 2.8625. **3.** 8.9218 − 10. **4.** 1.5474. **6.** 7.7563 − 10. **7.** 0.7583. **8.** 2.4688. **9.** 6.8503 − 10. **11.** 8.9659 − 20. **12.** 20.3651. **13.** 167.0. **14.** 2,313,000. **16.** 0.001942. **17.** 0.1401. **18.** 5.908. **19.** 0.01456. **21.** 2.063×10^{-8}. **22.** 3.107×10^{-10}. **23.** 6.521×10^{10}. **24.** 1.030×10^{15}.

EXERCISE 68. *Page 250*

1. 22.89. **2.** 0.03199. **3.** 50,220. **4.** 3.768. **6.** 1910. **7.** 1.644×10^{9}. **8.** 0.5358. **9.** 18,220. **11.** 0.09530. **12.** 47.47. **13.** 0.1109. **14.** 2.624×10^{7}. **16.** 1.725×10^{-8}. **17.** 0.0314. **18.** 72.79. **19.** 9.314. **21.** 7.555. **22.** 1.529. **23.** 227,000. **24.** 4861. **26.** 1.063. **27.** -46.01. **28.** -16.05. **29.** 0.04336. **31.** 99.75. **32.** 15.01. **33.** 9.9894 − 10. **34.** 0.977.

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